STATISTICS OF THE SUM OF LOGNORMAL VARIABLES IN WIRELESS COMMUNICATIONS

Paulo Cardieri and Theodore S. Rappaport
Mobile and Portable Radio Research Group
Bradley Department of Electrical and Computer Engineering
432 New Engineering Building, Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0350 - e-mail: wireless@vt.edu

Abstract—Schwartz & Yeh's method and Wilkinson's method are widely used to compute the moments of the total co-channel interference in wireless communications, usually modeled as the sum of lognormal random variables. The accuracy of these methods has been studied in previous works, under the assumption of having all summands signals (individual interference signals) identically distributed. Such assumption rarely holds in practical cases of emerging wireless communications systems, where interference may stem from far-away macrocells and nearby transmitters, causing the interference signals to have different moments. In this paper we present an analysis of Wilkinson's method and Schwartz & Yeh's methods, for the general case when the summands have different mean values and standard deviations in decibel units. We show that Schwartz & Yeh's method provides better accuracy than Wilkinson's method and is virtually invariant with the difference of the mean values and standard deviations of the summands, and the number of summands.

I. INTRODUCTION

The sum of lognormal random variables (RVs) has several important applications in wireless communications. For example, in a cellular communications system, if we consider the effects of shadowing alone [1], the total co-channel interference signal received at a given location is usually modeled as the sum of lognormally distributed signals, transmitted from undesired co-channel base stations or mobiles. No exact expression for the distribution of the sum of lognormal RVs is known. However, it is well accepted that the distribution can be well approximated by another lognormal distribution. Several methods have been proposed for computing the moments of the resultant lognormal distribution [2]-[5], and Schwartz & Yeh's and Wilkinson's methods are most often used. While Wilkinson's and Schwartz and Yeh's methods allow the individual signals in the sum to have different mean values and standard deviations in decibel units, previous works have surprisingly assumed that all the summands have identical means and standard deviations. However, practical situations where the individual interference signals have different mean values and different standard deviations occur very often in wireless communications. Since the mean values of the interference signals depend on parameters such as the transmitter to receiver separation (T-R) distance and antenna gains, the area means of each signal differ significantly if the distances and the antenna patterns are significantly different. Furthermore, it is likely that each interference has a different standard deviation about the area mean, due to different physical shadow environments. A typical situation where interference signals have different area means and standard deviations occurs in indoor wireless communications systems in multifloored buildings. Measurements have shown [6], [7] that the standard deviation in decibel units of the signal received at a given location depends on the number of floors separating the transmitter and receiver. When analyzing the performance and capacity of wireless systems, the assumption that all interference signals have the same mean and standard deviation may be used for a first-order prediction. However, for more accurate capacity and performance predictions of emerging in-building and microcell wireless systems, a more accurate description of the statistics of the individual interference signals is required, by considering the appropriate values of mean and standard deviation of each individual interference signal.

In this paper, we present an accuracy analysis of Schwartz & Yeh's and Wilkinson's methods for the general case when the individual interference signals that compose the total co-channel interference have different mean values and different standard deviations. We show that the accuracy of Wilkinson's method, unlike for Schwartz & Yeh's method, is very sensitive to the difference between the mean values and standard deviations of the individual interference signals, and the number of signals in the sum.

The remainder of this paper is organized as follows. Section II briefly reviews Wilkinson's and Schwartz & Yeh's methods. Section III compares the means and standard deviations of the sum of lognormal signals computed using both methods, for a wide range of statistical distributions of the individual summands. Section IV concludes the paper.
II. SUM OF LOGNORMAL RANDOM VARIABLES

Consider that \( N \) interference signals arrive at the receiver from co-channel mobiles or base stations. Assuming that the effects of small scale fading are averaged out, the local mean power level \( I_i \) of the \( i \)-th signal undergoes lognormal variation, and, using decibel units, can be modeled as [1]

\[
X_i = 10\log_{10} I_i = m_{X_i} + \chi_i \quad \text{in dBm},
\]

where \( m_{X_i} \) is the area mean power in dBm. The term \( \chi_i \) is a zero-mean normally distributed RV in dB with standard deviation \( \sigma_{X_i} \), also in dB, due to the shadowing caused by large obstacles [1]. The area mean \( m_{X_i} \) is usually modeled as a function of the T-R separation \( d_i \), path loss exponent \( \gamma \), transmitted power \( P_{T,i} \), in dBm, and transmitter and receiver antenna gains \( G_{T,i} \) and \( G_{R,i} \), both in dB [1]

\[
m_{X_i} = P_{T,i} + G_{T,i} + G_{R,i} - 10K\gamma\log_{10} d_i.
\]

The constant \( K \) comprises all terms that do not change in the model. Under the reasonable assumption that the phase shift observed in each individual interference signal varies significantly due to scattering, such that the signals add incoherently (e.g., their powers add) when averaged over the local area, the total interference signal is modeled as the sum of \( N \) lognormally distributed signals

\[
I = \sum_{i=1}^{N} I_i.
\]

It is well accepted that the distribution of \( I \) can be approximated by another lognormal distribution [2]-[5], or, equivalently, that \( X = 10\log_{10} I \) follows a normal distribution. Wilkinson’s [2] and Schwartz & Yeh’s [3] methods then compute the moments of \( X \).

In order to simplify the derivation of both methods, we define the normal RV \( Y_i \) as

\[
Y_i = \ln I_i = \lambda X_i,
\]

with mean value \( m_{Y_i} = \lambda m_{X_i} \) and standard deviation \( \sigma_{Y_i} = \lambda \sigma_{X_i} \), where \( \lambda = \ln(10)/10 \).

Therefore, approximating the distribution of \( I \) by a lognormal distribution, we have

\[
I = e^{Y_1} + e^{Y_2} + \cdots + e^{Y_N} \approx e^Z = 10^X,
\]

where \( Z \) (in logarithmic units) and \( X \) (in dB) are normally distributed, and \( Z = \lambda X \). Wilkinson’s and Schwartz & Yeh’s methods compute the mean value and standard deviation of \( Z \) (\( m_Z \) and \( \sigma_Z \)) or \( X \) (\( m_X \) and \( \sigma_X \)) from the mean values and standard deviations of the summands \( Y_i \), as shown subsequently.

For generality, it is useful to assume that the individual signals \( I_i \) may be correlated to each other. This correlation may be due to the fact that shadowing loss is caused by large objects surrounding the mobiles or base stations. Therefore, even signals coming from different directions may be attenuated by the same obstacles, leading to correlation among the received signals. Let us define the correlation coefficient \( r_{ij} \) of \( Y_i \) and \( Y_j \) by

\[
r_{ij} = \frac{E[(Y_i - m_{Y_i})(Y_j - m_{Y_j})]}{\sigma_{Y_i}\sigma_{Y_j}}.
\]

A. Wilkinson’s Method

According to Wilkinson’s method, the mean value and standard deviation of \( Z \) in (5) are determined by matching the first and second moments of \( I \) with those of \( I_1 + I_2 + \cdots + I_N \):

\[
\begin{align*}
E(e^Z) &= E(e^{Y_1} + e^{Y_2} + \cdots + e^{Y_N}), \quad (7) \\
E(e^{2Z}) &= E((e^{Y_1} + e^{Y_2} + \cdots + e^{Y_N})^2). \quad (8)
\end{align*}
\]

The moments in (7) and (8) are evaluated by observing that, for a normal RV \( u \) with mean value \( m_u \) and variance \( \sigma_u^2 \), and any integer \( n \), we have [8]

\[
E(e^{nu}) = \exp(nm_u + \frac{1}{2}n^2\sigma_u^2). \quad (9)
\]

Therefore, using (9) in (7) and (8), we have

\[
\exp(m_Z + \sigma_Z^2/2) = \sum_{i=1}^{N} \exp(m_{Y_i} + \sigma_{Y_i}^2/2) = u_1, \quad (10)
\]

\[
\exp(2m_Z + 2\sigma_Z^2) = \sum_{i=1}^{N} \exp(2m_{Y_i} + 2\sigma_{Y_i}^2) + 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \exp(m_{Y_i} + m_{Y_j}) \times \exp \left[ \frac{1}{2}(\sigma_{Y_i}^2 + \sigma_{Y_j}^2 + 2r_{ij}\sigma_{Y_i}\sigma_{Y_j}) \right] = u_9, \quad (11)
\]

Solving (10) and (11) for \( m_Z \) and \( \sigma_Z \), and using \( Z = \lambda X \), we finally obtain

\[
m_X = (1/\lambda) (2\ln u_1 - \frac{1}{2}\ln u_2), \quad (12)
\]

\[
\sigma_X = (1/\lambda) \sqrt{\ln u_2 - 2\ln u_1}. \quad (13)
\]

B. Schwartz & Yeh’s Method

Schwartz and Yeh proposed a method based on the exact computation of the mean value \( m_X \) and standard deviation \( \sigma_X \) of the sum of \( N = 2 \) lognormal RVs. For \( N > 2 \), a recursive approach is used, approximating the sum of two lognormal RVs by another lognormal RV, and computing the mean and standard deviation of the sum. Consider the sum of \( N \) lognormal RVs in (5), rewritten as

\[
Z = \ln(e^{Z_1} + e^{Z_2} + \cdots + e^{Z_N}). \quad (14)
\]

Let \( Z_k \) denote \( \ln(e^{Z_{k-1}} + e^{Z_k}) \), where \( Z_{k-1} \) is assumed to be normally distributed. Schwartz & Yeh’s method
then computes the mean \( m_{Z_k} \) and standard deviation \( \sigma_{Z_k} \) of \( Z_k \), for \( k = 2, 3, \ldots, N \). Following notation used in [2], \( m_{Z_k} \) and \( \sigma_{Z_k} \) are given by

\[
m_{Z_k} = m_{Z_{k-1}} + G_1(m_{w_k}, \sigma_{w_k}) \tag{15}
\]

\[
\sigma_{Z_k}^2 = \sigma_{Z_{k-1}}^2 - 2r(\sigma_{Z_{k-1}})(\sigma_{Y_{k-1}} + \sigma_{Z_{k-1}}) \times
\]

\[
G_3(m_{w_k}, \sigma_{w_k}), \tag{16}
\]

where \( m_{w_k} \) and \( \sigma_{w_k} \) are the mean and standard deviation of \( w_k = Y_k - Z_{k-1} \). Since \( Y_k \) is normally distributed and \( Z_{k-1} \) is assumed to be normally distributed, \( w_k \) is also assumed to be a normal RV, with probability density function \( p_{w_k}(w_k) \), and mean \( m_{w_k} \) and standard deviation \( \sigma_{w_k} \) given by

\[
m_{w_k} = m_{Y_k} - m_{Z_{k-1}}, \tag{17}
\]

\[
\sigma_{w_k} = \sqrt{\sigma_{Z_{k-1}}^2 + 2r(\sigma_{Z_{k-1}})(\sigma_{Y_{k-1}} + \sigma_{Z_{k-1}})} \times
\]

\[
G_3(m_{w_k}, \sigma_{w_k}). \tag{18}
\]

The term \( r(\sigma_{Z_{k-1}})(\sigma_{Y_{k-1}} + \sigma_{Z_{k-1}}) \) in (16) and (18) is the correlation coefficient of \( Z_{k-1} \) and \( Y_k \), given by [5]

\[
r(\sigma_{Z_{k-1}})(\sigma_{Y_{k-1}} + \sigma_{Z_{k-1}}) = \frac{\sigma_{Z_{k-1}}}{\sigma_{Z_{k-1}}} + \frac{r(k-1)(\sigma_{Y_{k-1}} + \sigma_{Z_{k-1}})}{\sigma_{Z_{k-1}}} \times
\]

\[
x \times G_3(m_{w_k}, \sigma_{w_k}). \tag{19}
\]

The functions \( G_1, G_2 \) and \( G_3 \) in (15), (16) and (19) are given by

\[
G_1(m_{w_k}, \sigma_{w_k}) = E\{\ln(1 + e^{w_k})\}, \tag{20}
\]

\[
G_2(m_{w_k}, \sigma_{w_k}) = E\{\ln^2(1 + e^{w_k})\}, \tag{21}
\]

\[
G_3(m_{w_k}, \sigma_{w_k}) = E\{w_k - m_{w_k}\}\ln(1 + e^{w_k}). \tag{22}
\]

and must be numerically evaluated. Note that \( Z_1 = Y_1 \). By recursively applying expressions (15) through (22), for \( k = 2, 3, \ldots, N \), \( m_X \) and \( \sigma_X \) are finally computed as \( m_X = m_{Z_N}/\lambda \) and \( \sigma_X = \sigma_{Z_N}/\lambda \).

C. Monte Carlo Simulation

The accuracy analysis of Wilkinson's and Schwartz & Yeh's methods is carried out by comparing the mean values and standard deviations of the sum computed using these two methods, to the mean value \( m_{\text{sim}} \) and standard deviation \( \sigma_{\text{sim}} \) of the sum computed using extensive Monte Carlo simulation. The Monte Carlo simulation is based on generating \( M \) (where \( M > 10000 \)) sets of samples of \( N \) lognormally distributed signals with specific means \( m_X \), and standard deviations \( \sigma_X \), in decibel units, and correlated with correlation coefficient \( r_{ij} \). The absolute errors in decibel units in the mean value and standard deviation for both methods are then determined as

\[
\text{error in the mean} = |m_X - m_{\text{sim}}| \tag{23}
\]

\[
\text{error in the standard deviation} = |\sigma_X - \sigma_{\text{sim}}| \tag{24}
\]

where \( m_X \) and \( \sigma_X \) are the mean and standard deviation of the sum \( X \), computed by both methods.

In the following, several cases of the sum of lognormal RVs are analyzed for uncorrelated and correlated signals, where the correlation coefficient between signals \( X_i \) and \( X_j \) is set to \( r_{ij} = 0.7 \), for the correlated case.

III. COMPARISON

A. Two summands with different mean values and standard deviations

Consider the sum of two lognormal RVs \( I_1 \) and \( I_2 \). The mean value and standard deviation of \( X_1 = 10\log_{10} I_1 \) are set to \( m_{X_1} = 0 \) dBm and \( \sigma_{X_1} = 8 \) dB, respectively, while the mean value and standard deviation of \( X_2 = 10\log_{10} I_2 \) are chosen such that \( \Delta m = m_{X_2} - m_X \), and \( \Delta \sigma = \sigma_{X_2} - \sigma_X \), are within the ranges \((-40 \text{ dB} \leq \Delta m \leq 40 \text{ dB}) \) and \((-4 \text{ dB} \leq \Delta \sigma \leq 4 \text{ dB}) \).

The errors in the mean and standard deviation for uncorrelated \( I_1 \) and \( I_2 \), using Wilkinson's method, are shown in Figure 1. For Wilkinson's method, when one of the summands is dominant \((|\Delta m| > 20 \text{ dB}) \), the errors in the mean and standard deviation are negligible. However, when the summands have about the same mean value and different standard deviations, the errors in the mean value and standard deviation of \( X \) are not negligible and can be as high as 6 dB and 4 dB, respectively, for uncorrelated summands. For correlated summands, additional results show that the maximum errors in the mean and standard deviation drop to 4 dB and 2 dB, respectively. On the other hand, the errors for Schwartz & Yeh's method can be shown to be zero, for both correlated and uncorrelated summands, since this method computes the exact mean and standard deviation of \( X \).

The higher accuracy of Schwartz & Yeh's method, compared with Wilkinson's method, is at the expense of higher complexity, as one can see from expressions presented in Section II. However, Figure 1 shows that, for some values of \( \Delta m \) and \( \Delta \sigma \), Wilkinson's method presents accuracy that may be acceptable for some applications. Let us assume that the maximum tolerable.
able error between the true value and the analytical approach is 1 dB, in both mean and standard deviation. We can show that, if the difference between the mean values of two uncorrelated signals $X_1$ and $X_2$ is less than 10 dB, the errors in both mean value and standard deviation computed using Wilkinson's method are larger than 1 dB, regardless of the difference between the standard deviations. For correlated signals with correlation coefficient of 0.7, the errors in the mean value and standard deviation are larger than 1 dB if the difference between the mean values is less than 10 dB and the difference between the standard deviations is larger than 2 dB.

B. $N$ summands with different mean values and same standard deviation

Now consider the case of the sum of $N$ (2 ≤ $N$ ≤ 18) lognormal RVs with the same standard deviation $\sigma_X = 8$ dB, but different mean values. The mean values $m_X$ of the $N$ summands are equally spaced distributed over the interval from $(0 \text{ dBm} - \delta m/2)$ to $(0 \text{ dBm} + \delta m/2)$, where $\delta m$ is the width of the interval and is adjusted from 0 to 50 dB in our simulation.

Figure 2 shows that the errors in $m_X$ and $\sigma_X$, computed using Wilkinson's method for uncorrelated summands, are non-zero over the entire ranges of $\delta m$ and $N$ considered. The manner in which the errors vary with $\delta m$ depends on the number of summands $N$. For small $N$ ($N < 8$), both errors tend to decrease as $\delta m$ increases. This is due to the fact that, with few summands and large $\delta m$, the summand with the largest mean value dominates the sum and the other summands are negligible, reducing the errors. However, when the number of summands is large ($N > 10$), the errors in $m_X$ and $\sigma_X$ increase as $\delta m$ increases. For large $N$, several summands will have large mean values, giving rise to the conclusion that the accuracy of Wilkinson's method degrades as the number of summands with about the same mean value increases. We see that the errors in the mean and standard deviation in Wilkinson's method are larger than 1 dB in almost all the ranges considered, except when the number of summands is small ($N < 3$) and $\delta m > 10$ dB. For $N$ correlated RVs, with correlation $r_{ij} = 0.7$, it can be shown that the errors in $m_X$ and $\sigma_X$ are almost invariant with $N$ and $\delta m$, and are smaller than 0.3 dB.

Comparisons between our simulations and values of $m_X$ and $\sigma_X$ for Schwartz & Yeh's method demonstrate that errors are almost invariant with $N$ and $\delta m$, and smaller than 0.5 dB, for both uncorrelated and correlated signals.

C. $N$ summands with different standard deviations and same mean value

Consider now the case of the sum of $N$ (2 ≤ $N$ ≤ 18) lognormal RVs with the same mean value $m_X = 0$ dBm, but different standard deviations. The standard deviations $\sigma_X$ of the $N$ summands are equally spaced distributed over the interval from $(8 \text{ dB} - \delta \sigma/2)$ to $(8 \text{ dB} + \delta \sigma/2)$, where $\delta \sigma$ can be adjusted from 0 to 8 dB.

The errors in the mean and standard deviation for Wilkinson's method, with respect to Monte Carlo simulation results, are presented in Figure 3 for uncorrelated signals. The results for Wilkinson's method agree with the well known results that the accuracy of Wilkinson's method degrades as the standard deviations of the individual signals increase [3]. Figure 3 also shows that, for uncorrelated signals, the errors are larger than 1 dB over the entire range analyzed.

Additional results show that, for correlated signals, the errors in the mean and standard deviation vary with $N$ and $\delta \sigma$ in the same manner as for uncorrelated signals. It can be shown that, for correlated signals, the errors in the mean and standard deviation may be acceptable (smaller than 1 dB), if $N$ and $\delta \sigma$ meet the following conditions:

- Error in the mean < 1 dB if $\delta \sigma < 0.15N + 1.72$ (dB)
- Error in the standard deviation < 1 dB if $\delta \sigma < 0.14N + 2.33$ (dB)

Our results also show that the errors in the mean and standard deviation in Schwartz & Yeh's method are virtually zero and invariant with the number of signals $N$ and range $\delta \sigma$. 
D. Summands with different mean values and standard deviations

Now consider the case of \( N = 6 \) summands, with different mean values and different standard deviations. The standard deviations of the summands are equally spaced distributed over the interval from \((8 \text{ dB} - \delta \sigma / 2)\) to \((8 \text{ dB} + \delta \sigma / 2)\), and \(\delta \sigma\) can be adjusted from 0 to 8 dB. Likewise, the mean values are equally spaced distributed over the interval from \((0 \text{ dB} - \delta m / 2)\) to \((0 \text{ dB} + \delta m / 2)\), where \(\delta m\) can be adjusted from -75 to 75 dB. Note that we allow \(\delta m\) to be negative in order to analyze not only the effects of the spreads \(\delta m\) and \(\delta \sigma\) on the accuracy of both methods, but also the effects of

- summands with small (large) mean values having small (large) standard deviations, which is obtained by using large \(\delta \sigma\) and large positive \(\delta m\),
- summands with small (large) mean values having large (small) standard deviations, which is obtained by using large \(\delta \sigma\) and large negative \(\delta m\).

The errors for Wilkinson's method are shown in Figure 4 for uncorrelated signals. Additional results show that for correlated signals, the errors vary in the same manner as in the uncorrelated case, but the maximum errors in the mean and standard deviation drop to 5 and 3 dB, respectively. The impression of the estimation of mean and standard deviation using Wilkinson's method increases as the spread of standard deviations of the summands increases (large \(\delta \sigma\)), and the spread of mean values decreases (small \(\delta m\)), for both uncorrelated and correlated signals. We can show that, for uncorrelated signals, the errors are only acceptable (smaller than 1 dB) when signals with small (large) mean values have large (small) standard deviations. For correlated signals, additional results show that the errors are not acceptable only when all signals have similar mean values and the spread of the standard deviations is large.

On the other hand, additional results show that the errors in both mean and standard deviation in Schwartz & Yeh's method are virtually zero, for both uncorrelated and correlated signals.

IV. Conclusion

Schwartz & Yeh's and Wilkinson's methods are widely used for computing the moments of the sum of lognormal distributed signals, such as in the analysis of co-channel interference in wireless communications systems. These two methods have been extensively analyzed in previous studies under the assumption of having all the summands identically distributed. This assumption rarely holds in practical cases of wireless communications systems, where interference signals coming from different physical environments may present different mean values and standard deviations in decibel units. To understand the conditions needed to accurately use both methods, we developed an extensive simulation to compare the performance of Schwartz & Yeh's and Wilkinson's methods for the general case, when the summands have different means and different standard deviations.

In all cases analyzed, Schwartz & Yeh's method presented excellent accuracy, for both uncorrelated and correlated signals. However, from the results presented in this work, we have shown quantitative results which highlight the fact that the accuracy of Wilkinson's method degrades as the spread of mean values of the summands decreases and as the spread of the standard deviations of the summands increases. It is also observed that the performance of Wilkinson's method degrades as the correlation of the signals in the summation decreases, and the number of summands increases. Such results may be useful in simulation and analysis of co-channel signals in emerging wireless communications systems.

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