A signal is defined as any physical quantity that varies with time, space or any other independent variable or variables.

- Continuous time signal e.g. current in a circuit
- Discrete time signal e.g. the variation in the number of shoes sold in a shop during a month

The discrete time signal arise due to the sampling of continuous signal. Selecting values of an analog signal at discrete-time instants is called sampling. All measuring instruments that take measurements at a regular interval of time provide discrete-time signal.

In general, if the continuous time signal is represented as \( x(t) = f(t) \), then the discrete-time signal, which is obtained by sampling the continuous time signal every \( T \) seconds is represented as \( x[n] = f(nT) \).

Quantization of discrete signals gives rise to digital signals or discrete valued signals. A quantized signal assumes only discrete amplitude values.

In this signal, both the amplitude and time variables can take only certain values.
A system may be defined as a physical device that performs an operation on a signal. It can also be regarded as a process that transforms one signal into another.

### Properties of Signals

**Periodic**

\[ x(t) = x(t+KT) \]

\[ x[n] = x[n+KN] \]

**Even**

\[ x(-t) = x(t) \]

\[ x[-n] = x[n] \]

**Odd**

\[ x(-t) = -x(t) \]

\[ x[-n] = -x[n] \]

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**Basic Discrete-time signals**

**Exponential Signals**

\[ x[n] = C \alpha^n = C e^{\beta n} \]

**Sinusoidal Signals**

\[ x[n] = A \cos(\omega n + \phi) \]

**Unit Impulse**

\[ \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \]

**Unit Step**

\[ u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} \]

**Rectangular Pulse**

\[ p[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \]

**Ramp Function**

\[ r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases} \]
Properties of Systems

- Systems with and without memory
  Memoryless: the output at a given time is dependent only at that same time
  e.g. \( y[n] = (2x[n] - x'[n]) \) input-output relationship of a resistor
  e.g. digital k-map
  with memory: e.g. \( y[n] = x[n-1] \) \( y[n] = y[n-1] + x[n] \)
  RC RL circuit, flip flop (store charge/voltage) (store previous states)

- Causality
  Causal: the output at any time depends on values at only the present and past times,
  e.g. \( y[n] = x[n-1] + x[n] \)
  Noncausal: the output depends on future inputs
  e.g. \( y[n] = x[n] - x[n+1] \) image processing, speech processing (recorded previously)

- Stability
  stable: if the input is bounded, the output must also be bounded (do not diverge)
  e.g. \( y(n) = \sin(2n) \)
  unstable: the output is not bounded
  e.g. \( y(n) = n^2 \) \( y[n] = \sum_{k=-\infty}^{n} [y[k]] \)

- Time Invariance
  time invariant: if a time shift in the input signal results in an identical time shift in the output signal
  e.g. \( y(t) = \sin(\pi t) \rightarrow y(t-t_0) = \sin(\pi (t-t_0)) \)
  time varying:
  e.g. \( y[n] = n x[n] \rightarrow y[n] = n \delta[n-1] \) \( y[n] = n \delta[n-1] \neq (n-1) \delta[n-1] \)

- Linearity
  Linear: if the system possesses the property of superposition. (Additivity + Scaling)
  \( x_i(t) \rightarrow y_i(t) \) \( x_i(t) \rightarrow y_i(t) \)
  \[ a x_i(t) + b y_i(t) \rightarrow a y_i(t) + b y_i(t) \]
  linear combination of inputs linear combination of outputs
  e.g. \( y(t) = t x(t) \) \( y[n] = 2x[n] \)
  Nonlinear: e.g. \( y(t) = x(t) \) \( y[n] = 2x[n] + 3 \) use zero input leads to zero output
LTI Systems

Many physical processes can be modeled as linear time-invariant (LTI) systems.

One of the primary reasons LTI systems are amenable to analysis is that any such system possesses the superposition property. As a consequence, if we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can use superposition to compute the output of the system in terms of its responses to these basic signals.

One of the important characteristics of the unit impulse is that very general signals can be represented as linear combinations of delayed impulses. This will allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse, e.g., RLC are all linear and make up LTI.

**Representation of Discrete-time Signals in terms of Impulse**

A discrete-time signal \( x[n] \) \( \Rightarrow \) A sequence of individual impulses.

\[
\begin{align*}
&x[-2] \delta[n+2] = \begin{cases} x[-2] & n = -2 \\ 0 & n \neq -2 \end{cases} \\
&x[-1] \delta[n+1] = \begin{cases} x[-1] & n = -1 \\ 0 & n \neq -1 \end{cases} \\
&x[0] \delta[n] = \begin{cases} x[0] & n = 0 \\ 0 & n \neq 0 \end{cases} \\
&x[1] \delta[n-1] = \begin{cases} x[1] & n = 1 \\ 0 & n \neq 1 \end{cases} \\
&x[2] \delta[n-2] = \begin{cases} x[2] & n = 2 \\ 0 & n \neq 2 \end{cases}
\end{align*}
\]

\[
x[n] = x[-2] \delta[n+2] + x[-1] \delta[n+1] + x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2]
\]

\[
x[n] = \sum_{k=0}^{\infty} x[k] \delta[n-k]
\]

represent \( x[n] \) as a superposition of scaled and shifted impulses

linear combination of shifted unit responses \( \delta[n-k] \), with weight \( x[k] \)

e.g., \( u[n] = \sum_{k=0}^{\infty} \delta[n-k] \cdot a[k] \delta[n-1] \)
Discrete-time Impulse Response

The response of a linear system to input signal \( x[n] \) will be the superposition of scaled responses of the system to each shifted impulses.

The response of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another.

\[
\begin{align*}
    x[n] &= \sum_{k=-\infty}^{\infty} x[n-k] h[k] \\
    y[n] &= \sum_{k=-\infty}^{\infty} x[n-k] h_k[n]
\end{align*}
\]

\( h_k[n] \): response to shifted unit impulse \( \delta[n-k] \)

\( y[n] \): scaled time-shifted response

If we know the response to the set of shifted unit impulses, we can construct the response to an arbitrary input.

\[ e.g. \quad x[n] \quad h_k[n] \quad y[n] \]

\[
\begin{align*}
    &\begin{array}{c|c|c|c|}
    n & -1 & 0 & 1 \\
    \hline
    x[n] & \begin{array}{c}
    1
    \end{array} & \begin{array}{c}
    1
    \end{array} & \begin{array}{c}
    1
    \end{array} \\
    h_k[n] & \begin{array}{c|c|c}
    1 & 0 & 1 \\
    \hline
    \end{array} & \begin{array}{c|c|c}
    1 & 0 & 1 \\
    \hline
    \end{array} & \begin{array}{c|c|c}
    1 & 0 & 1 \\
    \hline
    \end{array} \\
    y[n] & \begin{array}{c|c|c}
    1 & 1 & 1 \\
    \hline
    \end{array} & \begin{array}{c|c|c}
    1 & 1 & 1 \\
    \hline
    \end{array} & \begin{array}{c|c|c}
    1 & 1 & 1 \\
    \hline
    \end{array}
    \end{array}
\end{align*}
\]

If the linear system is time invariant, \( h_k[n] \) is a time-shifted version of \( h_0[n] \).

\[
h_k[n] = h_0[n-k]
\]

We define \( h[n] = h_0[n] \) \( \leftarrow \) unit impulse response (output of an LTI system when input is \( \delta[n] \)).

\[
y[n] = \sum_{k=-\infty}^{\infty} y[k] h[n-k] = x[n] * h[n]
\]

\( \text{Convolution sum/superposition sum} \)
The response of an LTI system to an arbitrary input is expressed as the superposition of the system's response to the unit impulse. The response due to input $x[k]$ applied at time $k$ is $x[k]h[n-k]$ (a shifted and scaled version of $h[n]$). The actual output is the superposition of all these responses.

### Example 1

**Method 1**

$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

$y[n] = x[0]h[n] + x[1]h[n-1]$

$y[n] = 0.5h[n] + 2h[n-1]$

By considering the effect of superposition in each individual output sample, we have another way to visualize the calculation of $y[n]$ using convolution.

### Example 1

**Method 2**

- $y[n] = 0$ when $n < 0$
- $y[0] = \sum_{k=-\infty}^{\infty} x[k] h[0-k] = 0.5$
- $y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1-k] = 0.5 + 2 = 2.5$
- $y[2] = \sum_{k=-\infty}^{\infty} x[k] h[2-k] = 0.5 + 2 = 2.5$
- $y[3] = \sum_{k=-\infty}^{\infty} x[k] h[3-k] = 2$
- $y[n] = 0$ when $n > 3$
Properties of LTI Systems

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \ast h[n] \]

- **Commutative Property**
  \[ x[n] \ast h[n] = h[n] \ast x[n] \]

- **Distributive Property**
  \[ x[n] \ast (h_1[n] + h_2[n]) = x[n] \ast h_1[n] + x[n] \ast h_2[n] \]

- **Associative Property**
  \[ x[n] \ast (h_1[n] \ast h_2[n]) = (x[n] \ast h_1[n]) \ast h_2[n] \]

**Example**
\[ x[n] = \begin{cases} u[n] + 2^n u[-n] & \text{for } n \geq 0 \\ \frac{1}{n+1} & \text{for } n < 0 \end{cases} \]
\[ h[n] = u[n] \]
\[ y[n] = (x[n] + x[n]) \ast h[n] \]
\[ = x[n] \ast h[n] + x[n] \ast h[n] \]