Signals and systems that process them exist in everyday life and in many applications that we commonly think of, and also ones that we don't.

**Communications**, **cell phones**, **Sonar**, **Weather Forecasts**, **Biomedical Engineering**, **Physical Sciences**, **Multimedia**, **Control Systems**

From our previous lecture, we know that signals can be either continuous-time signals (Independent Variable is a continuous variable), or a discrete-time signal (Independent Variable is an integer value).

**Implementation Platforms**:
- **Continuous Time (CT) hardware**: circuits, electronics
- **Digital Hardware**: microprocessors, ASICs, FPGAs
- **Digital Software implemented on general purpose digital systems**: C++, Java, MATLAB
- **Digital Software implemented on microprocessors (DSP chips)**: assembly language, cross-compiled C programs

We will deal with a specific class of systems: Linear, Time-Invariant Systems (LTI) and will look at signal representations in the **time domain** and **frequency domain**, learn relationships between the two, and how to convert one to another.

**Signals**

**Continuous-Time (CT)**

\[ x(t) \]

Images are two-dimensional:
- Brightness (horizontal, vertical)

**Discrete Time (DT)**

\[ x[n] \]

Also multi-dimensional signals \[ x[m,n] \]


Systems

\[ x[n] \quad \xrightarrow{\text{System}} \quad y[n] \]

LTI

linear \quad \text{time-invariant}

Interconnections of signals

Parallel, Serial, or Feedback

Time Domain

\[ x[t] \quad \Rightarrow \quad y[t] \]

Frequency Domain

Fourier Transform

Laplace Transform

Z-Transform

- Periodic signals \( x[t] \Rightarrow \) mathematically defined as a function \( f(t) = f(t+T) \) \( T \) = period

- Causal signals \( x[t] \Rightarrow f(t) \) defined as \( f(t) = 0 \) for \( t < 0 \), signal begins at \( t = 0 \)

- Non-causal signals \( f(t) \) 

- "Anti-causal" signals defined as \( f(t) = 0 \) only for \( t > 0 \), opposite of causal

All anti-causal systems are non-causal by definition.

What about periodic signals? Are they causal, non-causal, or anti-causal?

Periodic signals never end and are infinite length.
Stability

$|f(t)| \leq \infty, \forall t$

$|f[n]| \leq \infty, \forall n$

$f(t) = e^{-t}$ non-stable, $t \to \infty$

$f(t) = \frac{1}{t}$ stable

$f(t) = \cos(t)$, stable

$f(t) = \sin(t)$, stable

Quantity Signals

Energy = Power over time

Instantaneous signal power: $P(t) = |x(t)|^2$

Signal Energy: $E(t) = \int_{-\infty}^{\infty} |x(t)|^2 dt$

Average Signal Power:

$P(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} |x(t)|^2 dt$

$P(\tau_0, \infty) = \frac{1}{\tau_0} \sum_{n=\tau_0}^{\infty} |x[n]|^2$

Usual limits are over an infinite interval:

$E[0] = \sum_{-\infty}^{\infty} |x[n]|^2$

$E[0] = \sum_{n=-\infty}^{\infty} |x[n]|^2$

$P[0] = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^2 dt$

$P[0] = \lim_{T \to \infty} \frac{1}{T} \sum_{n=-T}^{T} |x[n]|^2$

$P[0] = \lim_{T \to \infty} \frac{1}{T} \sum_{n=-T}^{T} |x[n]|^2$
• Some signals have infinite power, energy, or both.
  - Energy signals if \( E_{oo} < \infty \)
  - Power signal if \( 0 < E_{oo} < \infty \)
• A signal can be an energy signal, a power signal, or neither.
• A signal cannot be both an energy and a power signal.
• If \( E_{oo} = \infty \) \( \Rightarrow P_{oo} = 0 \)
• Signals with finite average power have infinite energy \( P_{oo} > 0 \Rightarrow E_{oo} = \infty \)

**Signal Transformations**

**Two Main Concepts**
- Signals may be represented as a linear combination of basic signals
- Many systems can be described and analyzed in terms of their responses to basic signals.

**Basic Signals**

**Discrete Time Signals**
- Impulse (or unit impulse)
  - \( \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \)
- Step Function
  - \( u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \)
- Ramp Function
  - \( r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases} \)
**Pulse (of length N)**

\[ p[n] = \begin{cases} 1 & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases} \]

**Euler's Identities** For real and complex sines/cosines:

- \( e^{jx} = \cos x + j \sin x \)
- \( \cos x = \frac{1}{2} e^{jx} + \frac{1}{2} e^{-jx} \)
- \( \sin x = \frac{1}{2j} e^{jx} - \frac{1}{2j} e^{-jx} \)

**Basic Signal Operators**

- **Time Shift (Delay)**: \( x[n-t] + x[n-m] \)
  - If \( m > 0 \), shift to right
  - If \( m < 0 \), shift to left

**Example**

- If \( u[n] \), plot \( x[n] = u[n-m] \) for \( m = 3 \)

\[ u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad u[n-m] = \begin{cases} 1 & n-m \geq 0 \\ 0 & n-m < 0 \end{cases} \]

**Fold/Time Reversal** => a reflection about \( n = \beta \)

- Plot \( x[\bar{n}] = u[\bar{n}] \)

- \( x[\bar{n}] = u[\bar{n}] \) or "right sided"

- \( x[\bar{n}] = u[\bar{n}]-u[\bar{n}-1] \) is "left sided"

**Time Scaling** \( x[\alpha n] \)

- \( \alpha > 1 \), signal compresses
- \( \alpha < 1 \), signal stretches

- If \( \alpha = 1 \), signal stays the same
Basic Properties can be combined!

**Continuous Time Signals**

- **Step**
  \[ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]

- **Ramp**
  \[ r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \]

- **Pulse**
  \[ p(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \]

- **Impulse**
  \[ \delta(t) \text{, impulse of unit area} \]
  \[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

**Similar Operators**

- **Shift (delay)**
  \[ x(t) \rightarrow x(t-a) \]

- **Fold**
  \[ x(t) \rightarrow x(-t) \]

- **Fold+Shift**
  \[ x(t) \rightarrow x(t+r) \]

**Multiplication + Addition: Point-by-Point in-time**

**Differentiation + Integration**

\[ y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau = \int_{-\infty}^{\infty} x(t+\tau) \, d\tau \]

\[ \frac{d}{dt} [c \cdot x(t)] = c \cdot \dot{x}(t) \]

\[ \int_{-\infty}^{t} [c \cdot x(\tau)] \, d\tau = c \cdot \int_{-\infty}^{t} x(\tau) \, d\tau \]

**Even + Odd Signals**

- **Even**
  \[ x(t) = x(t) \]
  \[ x(t) = x_{e}(t) + x_{o}(t) \]

- **Odd**
  \[ x(t) = -x(t) \]
  \[ x(t) = \frac{1}{2} [x(t) + x(-t)] \]

\[ x_{e}(t) = \text{even signal} \]
\[ x_{o}(t) = \text{odd signal} \]

Any signal can be written as the sum of an odd + even signal.
**Linearity**

\[ x(t) \rightarrow [h(t)] \rightarrow y(t) \]

Say we have \( x_1(t) \rightarrow y_1(t) \)
\( x_2(t) \rightarrow y_2(t) \)

A system is linear only if

\[ a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t) \]

**Time Invariance**

A system is time invariant if and only if \( x[n] \rightarrow y[n] \)
implies that \( x[n-n_0] \rightarrow y[n-n_0] \)

Circuits with no energy stored are time invariant.

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**Impulse Response**

The impulse response of a linear system \( h(t) \) is the output of the system at time \( t \) to an impulse at time \( t_r \),

\[ h(t) = H(\delta(t)) \]

**Impulse as \( \delta(t) \):**

\[ \int_{-\infty}^{\infty} \delta(t-t_r) dt = \begin{cases} 1 & t = t_r \\ 0 & \text{otherwise} \end{cases} \]

\[ \int_{-\infty}^{b} g(t) \delta(t) dt = \int_{-\infty}^{b} g(t) \cdot 0 \, dt = 0 \]

**Sifting Property**

- Evaluate the integral as a function multiplied by an impulse at the origin,

\[ \int_{-\infty}^{\infty} g(t) \delta(t) dt = \int_{-\infty}^{\infty} g(t) \cdot 0 \, dt \]

because impulse is zero everywhere else

\[ \int_{-\infty}^{\infty} g(t) \delta(t) dt = 0 \]

\[ \int_{-\infty}^{\infty} g(t) \delta(t) dt = \int_{-\infty}^{\infty} g(t) \cdot f(t) \, dt \]

in general

\[ \int_{-\infty}^{b} g(t) \delta(t) dt = f(t_r) \int_{-\infty}^{b} g(t) dt \]

\[ f(t_r) = \begin{cases} 0 & \text{otherwise} \\ g(t) \end{cases} \]
Thus, because $g(t-T)$ is $\delta$ except at $T$, we can show:

$$\int_a^b g(t-T) f(t) \, dt = \begin{cases} f(T) & a < T < b \\ 0 & \text{otherwise} \end{cases}$$

This results in the shifting property:

$$\int_a^b g(t-T) x(t) \, dt = x(T)$$

### Discrete Time

**Discrete time: $x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$**

- **Discrete Time Convolution**
  
  Any discrete-time input signal $x[n]$ can be expressed as a sum of scaled unit impulses.

  $$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

  By linearity and time-invariance, the output of the system is the scaled sum of outputs due to each unit impulse.

  $$x[n] \rightarrow y[n] = h[n] \ast x[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

  \[\text{Convolution operator}\]

  **Discrete time convolution sum:**

  If we know $h[n]$, we can calculate $y[n]$ for any input $x[n]$.

  $h[n]$ completely characterizes the LTI system.

  **Examples:**

  1. Flip impulse response $h[k] \rightarrow h[-k] \rightarrow y[n]$

  2. Drag/shift $h[-k]$ through $x[k]$ over $n$, yielding $x[k-n] \cdot h[-(k-n)]$ and multiply pointwise by $x$, yielding $x[k] \cdot h[n-k]$ such that $y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$
Sampling + Quantization

- Analog signals are discretized through sampling. When sampled, they are then quantized into digital signals.

We convert from analog to digital. For storage, it must quantize to specific discrete levels to fit into a finite amount of memory. More quantized levels, the more fidelity a resolution we get.

Sampling converts \( x(t) \) to \( x(n) \), much like a \( y(t) = \text{filter operator} \)

\[
x[n] = x(nT_s)
\]

\( T_s = \frac{1}{50} \)
Sampling is a bridge between CT & DT.

How often should we sample in order to NOT lose any information?

If \( x(nT_s) = x(nT_s) \)

\[ T_s \]
interval between samples

\[ f_s = \frac{1}{T_s} \]
is sampling rate

Fourier Transform next week will help with the

N\text{yquist-Sampling} theorem that states we need to sample a signal at
twice its highest frequency \( f_s = 2f_{\text{signal}} \) to be able to

fully reconstruct it.

Sampling at the Nyquist Rate will allow us to remove aliasing artifacts from sampling.

After sampling, we quantize the samples into digital sets or quantities.

Quantization levels are usually \( N = 2^N \) where \( N \) is the number of
levels & \( R \) is the number of bits.

2-bit Quantizer: \( 2^2 = 4 \) levels

DAC = Digital-to-Analog Converter for
transmission

ADC = Analog-to-Digital Converter for
Sampling at discrete quantized levels.

If we sample a signal ranging from

0 to 4 volts with quantized bins of 0.5 volts,

we have 8 amplitude levels: 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4 volts

\[ 2^3 = 8 \] results in a 3-bit ADC/Quantizer
Karnaugh Map - Recap

\[ \overline{A} \overline{B} \overline{C} + \overline{A}BC + \overline{A}BC + \overline{A} \overline{B} \overline{C} \]

\( \text{\textsuperscript{\textdegree}} \text{s represent product term} \)

\[ AB + BC + BC \]

\[ AB + BC + AC \]

\[ f(A,B,C) = AB + BC + AC \]