Comparison of Conductive Heat Transfer and Diffusive Mass Transfer

1. Introduction. Previously, we have looked at three examples of heat conduction. Here we pose the corresponding problems for diffusive mass transfer, and transcribe the solutions obtained earlier for the heat conduction problems into solutions for the mass transfer problems. We recall that when mass transfer obeys Fick's Second Law of Diffusion,

$$\frac{\partial c_A}{\partial t} = D_{AB} \nabla^2 c_A \quad (c, D_{AB} \text{ constant; no bulk convection; no reactions}) \quad (1)$$

or

$$\frac{\partial \rho_A}{\partial t} = D_{AB} \nabla^2 \rho_A \quad (\rho, D_{AB} \text{ constant; no bulk convection; no reactions}) \quad (2)$$

the differential species balance is mathematically same in form as “unsteady”, or time-dependent, conduction,

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho \cdot \hat{c}_V}\right) \nabla^2 T \quad (\rho, k \text{ constant}; \mathbf{v} = 0; \dot{q} = 0) \quad (3)$$

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad (\rho, k \text{ constant}; \mathbf{v} = 0; \dot{q} = 0; \hat{c}_V = \hat{c}_T) \quad (4)$$

The equivalence in mathematical form between the mass transfer equations (1)-(2) and the heat transfer equations (3)-(4) indicate that solutions to such equations will be interchangeable.

The assumptions that apply to each of the above equations are listed next to them. Perhaps the most subtle one is the assumption of no bulk convection in equations (1) and (2). As discussed earlier, this not only means that there must be no forced or free convection, but also that the solutions are sufficiently dilute so that diffusion-engendered bulk convection is negligible. As discussed later in this handout, drawing analogies between mass and heat transfer has its limitations; on the other hand, sufficiently many situations arise in which such analogies are useful that a brief illustration of their application is warranted.

EXAMPLE 1. Steady-State, 1-Dimensional Heat Conduction or Mass Diffusion Through a Film. A film, of a uniform thickness $L$ in the vertical direction, consists of a quiescent (motionless) liquid of species $B$. We consider two situations:

(i) Heat Transfer (Fig. 1a). The bottom of the liquid film is in contact with a surface at a temperature $T_1$, while the top of the film is in contact with air at a temperature $T_2$. Also given:

(i) Constant thermal conductivity $k$

(ii) No generation of heat inside the film

(iii) Steady-state conditions
(iv) Liquid is motionless \((v = 0)\)
(v) The air is at a uniform temperature \(T_2\)

![Fig. 1a](image)

(ii) Mass Transfer (Fig. 1b). The bottom of the liquid film is in contact with solid \(A\) that is sparingly soluble in \(B\) with a saturation concentration \(c_A^S\). The top of the liquid film \(B\) is in contact with layer \(C\). \(B\) and \(C\) are immiscible. The partitioning coefficient of \(A\) in \(C\), relative to \(A\) in \(B\), is \(H_c\). Also given:

(i) Constant diffusion coefficient \(D_{AB}\)
(ii) No reactions involving \(A\)
(iii) Steady-state conditions
(iv) No forced or free convection
(v) \(A\) is sparingly soluble in \(B\)
(vi) Layer \(C\) is a well stirred with a uniform, constant concentration \(c_A^C\) of \(A\).

(a) Determine an expression for the temperature profile \(T(x)\) and the conductive heat flux \(q_{Fx}\) in the \(B\) layer for the heat transfer problem (Fig. 1a).

(b) Determine an expression for the concentration profile \(c_A(x)\) and the diffusive mass flux \(J_{Ax}\) across the \(B\) layer for the mass transfer problem (Fig. 1b).

Setting up and Solving the Heat Transfer Problem (Review). As discussed in the earlier handout on heat transport, this problem is described by the following ODE:

\[
\frac{d^2T}{dx^2} = 0
\]  

Equation (5) is a simplified equation (4), after making use of the steady state condition and the fact that \(T\) depends only on \(x\). We also have two boundary conditions on \(T\),

\[
T(0) = T_1
\]
\[
T(L) = T_2
\]

The solution for \(T\), derived previously, is

\[
T = [(T_2 - T_1)/L] x + T_1 \quad 0 < x < L
\]
Application of Fourier's Law,

\[ q_{Fx} = -k \frac{dT}{dx} \]  \hspace{1cm} (9)

then leads to the formula for the conductive heat flux across the B layer,

\[ q_{Fx} = -(k/L) (T_2 - T_1) \]  \hspace{1cm} (10)

Setting up the Mass Transfer Problem. We wish to derive the concentration profile of A solute inside the film of B solvent. We are given several simplifying assumptions. Perhaps most importantly, A is stated to be only sparingly soluble in B. This implies that the concentration of A in B will be always dilute. Low concentrations of A mean that the total concentration \( c \) will be everywhere nearly equal to that of pure B, and so \( c \) can be assumed constant to a very good approximation.

Furthermore, for such dilute conditions diffusion-engendered bulk convection becomes a minor effect. Combined with the absence of forced or free convection, bulk convection can then be neglected (\( \mathbf{V} = \mathbf{v} = 0 \)). Since both the molar and mass average velocities are approximately zero, there is no strong preference to use molar or mass units. Molar units will be used since the problem is stated in terms of molar concentrations \( c_A \) rather than mass concentrations \( \rho_A \).

We are also given that there are no reactions involving A and that \( D_{AB} \) is constant. These simplifications allow use of equation (1) to model the mass transport of A,

\[ \frac{\partial c_A}{\partial t} = D_{AB} \nabla^2 c_A \]  \hspace{1cm} (1)

Invoking the steady state condition (given), and the expectation that \( c_A \) will only depend on \( x \), leads to

\[ \frac{d^2 c_A}{dx^2} = 0 \]  \hspace{1cm} (11)

Equation (11) needs to be supplemented by two boundary conditions. It is customary to assume that local equilibrium holds across interfaces. Applied to the interface between liquid B and the solid A, this assumption implies that the concentration of dissolved A in B immediately at the interface will equal that which is in equilibrium with the solid A. At equilibrium, the concentration of A in the liquid would equal the saturation concentration \( c_A^S \). Therefore, assumption of local equilibrium produces the following boundary condition

\[ c_A(0) = c_A^S \]  \hspace{1cm} (12)
Similarly, at equilibrium the partitioning of $A$ between liquids $B$ and $C$ must obey the partitioning coefficient $H_c$. Since $H_c$ was defined as preference of $A$ for $C$ relative to that of $A$ for $B$, the concentrations of $A$ on the two sides of the interface between $B$ and $C$ are related by

$$\frac{c_A^C}{c_A^B} = H_c$$  \hspace{1cm} (13)

Equation (13) leads to the following boundary condition

$$c_A(L) = \frac{c_A^C}{H_c}$$  \hspace{1cm} (14)

c$_A^C$, $H_c$, and $c_A^S$ are listed as given in the problem statement. With the two boundary conditions (12) and (14) and the differential equation (11) the problem is now fully specified. Once $c_A(x)$ has been calculated, the diffusive flux of $A$ through the liquid $B$ layer would be obtained from Fick’s Law,

$$J_{Ax} = -D_{AB} \frac{dc_A}{dx}$$  \hspace{1cm} (15)

"Translating" the Heat Transfer Solution to the Mass Transfer Solution. In the two problems under consideration the unknowns of interest, $T$ and $c_A$, and $q_{Fx}$ and $J_{Ax}$, obey differential equations that are identical in form. We can verify this correspondence by direct, side by side comparison,

$$\frac{d^2 T}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 c_A}{dx^2} = 0$$  \hspace{1cm} (16)

$$q_{Fx} = -k \frac{dT}{dx} \quad \text{and} \quad J_{Ax} = -D_{AB} \frac{dc_A}{dx}$$  \hspace{1cm} (17)

The boundary conditions for both problems are also identical in that they specify the value of the unknown scalar field (i.e. $T$ or $c_A$) at $x = 0$ and at $x = L$,

$$T(0) = T_1 \quad \text{and} \quad c_A(0) = c_A^S$$  \hspace{1cm} (18)

$$T(L) = T_2 \quad \text{and} \quad c_A(L) = \frac{c_A^C}{H_c}$$  \hspace{1cm} (19)

By inspection of equations (16) to (19), performance of the following substitutions

$$T \rightarrow c_A$$
$$T_1 \rightarrow c_A^S$$
$$T_2 \rightarrow \frac{c_A^C}{H_c}$$
$$q_{Fx} \rightarrow J_{Ax}$$
$$k \rightarrow D_{AB}$$  \hspace{1cm} (20)
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would translate the heat transfer equations to become exactly the equations for the mass transfer problem. If the equations defining the problem are identical in this manner, so must be the solutions. Thus, knowing the solution for \( T \),

\[
T = [(T_2 - T_1)/L] x + T_1
\]  

(8)

we can directly write down the solution to the mass transfer problem by making the substitutions listed in (20). This leads to

\[
c_A = [(c_A C/H_c - c_A S)/L] x + c_A S
\]  

(21)

Similarly, we can directly "translate" the solution for \( q_{Fx} \),

\[
q_{Fx} = -k(T_2 - T_1)/L
\]  

(9)

to that for \( J_{Ax} \),

\[
J_{Ax} = -D_AB(c_A C/H_c - c_A S)/L
\]  

(22)

We have solved the mass transfer problem simply by recalling an earlier solution to the analogous problem from heat transfer. To do so, we had to carefully compare the differential equations and the boundary conditions from both problems to ensure that they are indeed equivalent and to learn what substitutions will "translate" the solution to the heat transfer problem to that for the mass transfer. Although solving the mass transfer problem directly would also have been straightforward in this case, for problems that are multi-dimensional in nature (e.g. requiring Separation of Variables) the ability to translate one solution to the other can be very helpful.

2. When Are Mass and Heat Transfer Problems "Analogous"? Perhaps the most important issue is why mass and heat transfer, in general, are not similar. Physically, transfer of heat and transfer of mass are very different processes. The first deals with transfer of energy and the second with transfer of material. Transfer of material is always accompanied by simultaneous transfer of energy (kinetic, potential, internal) possessed by that material; in contrast, energy can be transferred without a concomitant transfer of mass, such as by conduction or radiation. Thus, not only the quantities being transferred, but also the available modes of transfer, are different. The two processes, heat transfer and mass transfer, are therefore physically very distinct.

Nevertheless, as illustrated by Example 1, there are situations where the mathematical descriptions of mass and heat transfer are identical in form. Under what conditions does such correspondence arise? Below are some thoughts on this question, keeping in mind that each specific problem requires its own analysis to ensure that a correct analogy is being made. A mass transfer problem is most likely to have a physically relevant heat transfer analogue when:

- Diffusion-engendered bulk convection is negligible. Unlike diffusion of mass which engenders bulk convection of mass, conduction of heat does not engender bulk convection of heat. Therefore, heat transfer equations do not have a way to account for analogy to diffusion-engendered bulk convection.
Radiative heat transfer is negligible. In this mode of heat transfer, the heat flux is proportional to the 4th power of temperature of the body radiating the energy. There is no general process for mass transfer in which mass fluxes are generated in proportion to a concentration raised to the 4th power.

Chemical reaction terms, whether homogeneous \( r_A \) or \( R_A \) or heterogeneous, would have to have the same dependence on concentration as the heat generation terms in the analogous heat transfer problem have on temperature. In other words, the dependence of a homogeneous reaction rate \( r_A \) or \( R_A \) on the concentration \( c_A \) would need to be the same as that of the homogeneous heat generation \( \dot{q} \) on the temperature \( T \).

Convection, if present, should only involve flows that are incompressible and for which viscous dissipation is negligible. Convection is a mode of transport common to both mass and heat transfer, and thus may be possibly included when drawing analogies. However, if the flow is compressible one expects complications as performance of \( pV \) work, which is present in the internal energy equation for compressible flows, has no counterpart in the species mass balance. Dissipation of mechanical to internal energy, also present in the internal energy balance, is similarly absent from the species' mass balances. Thus compressible flows or flows in which significant viscous dissipation occurs make unlikely candidates for drawing heat-mass transfer analogies.

The above points are rough guidelines only, and each particular problem requires careful consideration. Let’s examine two more examples of mass-heat comparisons, based on previously studied heat conduction problems.

**EXAMPLE 2. Steady-State, 1-Dimensional Heat Conduction or Mass Diffusion in a Cylinder With a Homogeneous Source Term.**

Previously, we posed the following problem for heat transfer: A very long, metal cylinder of radius \( R \) is carrying electrical current (Fig. 2). Inside the cylinder, electrical energy is dissipated to heat by resistive heating, leading to a constant rate of heat production per volume \( \dot{q} \) (note: \( \dot{q} \) does not depend on position or time). The heat generated from the electrical dissipation is conducted away to the surroundings. Steady state conditions apply in the interior of the cylinder. The surface of the cylinder, at \( r = R \), is kept at a temperature \( T_R \). The heat conductivity of the metal cylinder is constant.

Determine the temperature distribution inside the cylinder.
The governing equations were derived to be (see earlier handout):

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{q}{k} = 0 \quad (23)$$

Equation (23) expresses the requirement that the heat conduction to a point (1st left term), plus the rate of heat generation at the point (2nd left term), must sum to zero. Physically, this reflects the constraint of steady state, under which accumulation of heat at a point must be zero. For steady state to be satisfied, the heat must be removed (by conduction) at the same rate as it is generated by the resistive heating. Since two integrations are required with respect to \( r \), we also needed two boundary conditions. These were:

$$T(r = R) = T_R \quad (24)$$

$$T(r = 0) \text{ remains finite (i.e. does not diverge to infinity)} \quad (25)$$

Boundary condition (24) comes directly from the problem statement. Boundary condition (25) was deduced earlier during the process of solving equation (23). In particular, (25) came from the need to eliminate a term \( C \ln(r) \) from the solution for \( T \), since this term diverges at \( r = 0 \). This was accomplished by setting the integration constant \( C = 0 \). Solution of (23) subject to boundary conditions (24) and (25) led to

$$T = T_R + \left( \frac{q}{4k} \right) (R^2 - r^2) \quad (26)$$

Given the above information:

(a) What would the analogous problem for mass transfer be? What would a schematic diagram of the problem look like?

(b) What would the governing equations (differential balance and boundary conditions) be for the mass transfer problem?

(c) What substitutions would be needed to translate equation (26) to the solution for the mass transfer problem?
In retrospect, Example 2 serves to illustrate some of the limitations in making analogies between heat and mass transfer. Whereas the heat generation term in Example 2 is perfectly physical, representing the resistive heating of a current-carrying wire, the meaning of the mass generation term is less obvious. In particular, its lack of dependence on concentration of \( A \) seems somewhat simplistic. Still, it could be a useful model for some situations. For example, imagine a cylindrical gel consisting of crosslinked polymer chains that carry attached drug molecules (bound solute species \( A \)), as in Fig. 3. The pores in the gel are filled with water (solvent \( B \)). If the bound solute species \( A \) is released by the polymer into the solvent \( B \) at a relatively constant rate (this may be a good approximation over durations during which the polymer is not significantly depleted of the drug), according to the irreversible reaction

\[
\text{Bound } A \text{ (attached to the polymer)} \rightarrow \text{Dissolved } A \text{ (in the } B \text{ solvent)}
\]

then effectively a constant, homogeneous reaction source of free (dissolved) \( A \) exists inside the gel. If steady state has been reached, then the dissolved \( A \) must diffuse out of the gel at the same rate as it is being released from its bound form on the polymer. The solution derived above should be a good model for the concentration profile of the dissolved \( A \) in the gel under these conditions (steady state; constant rate of release of bound \( A \) from the polymer).
EXAMPLE 3. Two-dimensional, Steady-State Heat Conduction or Mass Diffusion in a Thin Plate. In the Heat Conduction handout, following problem was used to illustrate the method of Separation of Variables:

An infinitely long (in the \(y\)-direction), very thin (in the \(z\)-direction) solid metal plate is maintained in contact with surroundings such that all of its edges are at a temperature \(T = 0\) except for the bottom edge which is kept at a temperature \(T_1\) (Fig. 4). In the \(x\)-direction, the plate has a width \(L\). There is no temperature variation along the \(z\)-direction, and the entire plate is kept under steady state conditions. The heat conductivity \(k\) of the plate is constant. Derive an expression for the temperature distribution in the plate.

![Diagram of heat conduction in a thin plate](image)

The governing PDE (a two-dimensional Laplace's equation) and boundary conditions for this heat transfer problem were

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\]

(27)

\(x\)-direction: \(T (0, y) = 0\) (left edge of plate) \(T (L, y) = 0\) (right edge of plate) \(T (x, 0) = T_1\) (bottom edge of plate) \(T (x, \infty) = 0\) ("top" edge of plate)

(28) \hspace{1cm} (29) \hspace{1cm} (30) \hspace{1cm} (31)

The solution, after several pages of calculations, was determined to be,

\[
T = \sum_{s=1}^{\infty} \frac{4T_1}{(2s-1)\pi} \sin((2s-1)\pi x/L) \exp(-(2s-1)\pi y/L)
\]

(32)

(a) In words, what would the analogous problem for mass transfer be? We need to pay particular attention to the definition and physical interpretation of boundary conditions. What would a schematic diagram of the problem look like?
(b) What would the governing equations (differential equation and boundary conditions) be for the mass transfer problem?

(c) What substitutions would be needed to translate the heat transfer solution (equation (32)) to the solution for the mass transfer version of this problem?