Integral and Differential Laws of Energy Conservation

1. State of Stress in a Flowing Fluid (Review). Recall that stress is force per area. Pressure exerted by a fluid on a surface is one example of stress (in this case, the stress is normal since pressure acts or pushes perpendicular to a surface).

At every point inside a flowing fluid there is stress. This stress can be conceptualized as follows. Imagine a small (more precisely, infinitesimal) “volume element” of fluid as shown in Fig. 1. The fluid element is embedded in a large sea of external fluid that exerts stress on the fluid element. The stress can arise, for example, from collisions and "rubbing" of the particles of this external fluid with those inside the element. Now consider the front face of the fluid element, which is oriented perpendicular to the $x_1$ axis. If we could measure the stress exerted by the external fluid on the fluid inside the element across the front face, we would in general be able to identify three different components of this stress: the normal stress $\sigma_{11}$, which acts perpendicular to the surface (see Fig. 1), and two shear stresses $\sigma_{12}$ and $\sigma_{13}$ that act tangential to the surface. The total force $F$ on the front face of the fluid element is, therefore

$$F = d\sigma_1 d\sigma_2 (\sigma_{11} d_1 + \sigma_{12} d_2 + \sigma_{13} d_3) \quad (1)$$

where $d\sigma_1 d\sigma_2$ is the total (differential) area of the front face. Similar expressions can be developed for the other five faces of the fluid element. In total, we would need nine components of the stress to fully specify the forces exerted on the different faces: $\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{21}, \sigma_{33}, \sigma_{32}, \sigma_{31}$. All nine components are illustrated in Fig. 1. We need nine stress components because for each of 3 possible orientation of a face (i.e. with the face oriented perpendicular to the $x_1, x_2, \text{ or } x_3$ axis), there are 3 independent components of the stress to consider (one normal and two shear), so we have $3 \times 3$ stress components in total. The nine stress components completely specify the so-called state of stress inside the fluid. Note that the stress components are function of position in the fluid, $\sigma_{ij} = \sigma_{ij}(x_1, x_2, x_3)$. We also recall that the $i,j$th component of the stress tensor, $\sigma_{ij}$, is the stress exerted in direction $j$ on a surface that is oriented perpendicular to direction $i$. You can verify this last statement by referring to Fig. 1.

The stresses $\sigma_{ij}$ arise from pressure exerted by the external fluid on that of the fluid element as well as viscous (frictional) "rubbing" between the external and internal fluid. Viscous stresses arise because fluid particles on the two sides of a face of the element travel at slightly different velocities; in other words, the average velocity of fluid immediately outside the element is different from that immediately inside. The difference in velocity causes the fluid particles to move relative to one another and, hence, produce "rubbing" that results in viscous frictional stresses. Viscous stresses are only present when the velocity changes from point to point inside a
fluid. Put another way, viscous stresses are directly related to the presence of velocity gradients. For a \textbf{Newtonian fluid}, the stress components are obtained from the following general expression (which, as expected, has velocity gradients in it):

\[
\sigma_{ij} = -p \delta_{ij} + \lambda (\nabla \cdot \mathbf{v}) \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{2}
\]

In equation (2), \(\delta_{ij}\) is the Kronecker delta, \(p\) is the thermodynamic pressure, \(\mu\) is \textbf{viscosity} of the Newtonian fluid, \(\lambda\) is called the \textbf{second coefficient of viscosity}, and \(v = v_1 \delta_1 + v_2 \delta_2 + v_3 \delta_3\) is the velocity of the fluid. \(\mu\) and \(\lambda\) are properties of the fluid. \(p\) in equation (2) is the true thermodynamic pressure (this is what we normally think of as pressure in a fluid at rest); however, in fluid mechanics it is very common to define a so-called \textbf{mechanical pressure} \(P = p - (\lambda + 2/3 \mu) (\nabla \cdot \mathbf{v})\) in order to simplify the form of equations (by defining the mechanical pressure \(\lambda\) is "hidden" in the equation of momentum conservation and does not appear explicitly). One often specializes to incompressible flows in which case there is no difference between the mechanical and thermodynamic pressures, \(P\) and \(p\), since \(\nabla \cdot \mathbf{v} = 0\) for incompressible flows. Therefore, for purposes of this course we will not be concerned with these somewhat subtle distinctions; however, should you at some point need to consider situations in which Newtonian fluids are greatly and rapidly compressed or expanded (large \(\nabla \cdot \mathbf{v}\)) you should bear in mind the difference between \(p\) and \(P\) and be careful to note which definition is being used. Equation (2) can be directly used to write down expression for the different normal and shear stresses, for example

\[
\sigma_1 = -p + \lambda \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) + 2\mu \frac{\partial v_1}{\partial x_1}
\]

and

\[
\sigma_2 = \mu \left( \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \right)
\]

Note that equation (2) implies that we are using the Cartesian coordinate system since in the last term on the right the derivatives are taken with respect to the Cartesian coordinate variables \(x_i\) (where \(i = 1, 2, \text{or } 3\)).

We could define a \textbf{stress tensor} \(\sigma\) as follows (for our present purpose we can think of this definition as just a notational convenience),

\[
\sigma = \sigma_{11} \delta_1 \delta_1 + \sigma_{12} \delta_1 \delta_2 + \sigma_{13} \delta_1 \delta_3 + \sigma_{21} \delta_2 \delta_1 + \sigma_{22} \delta_2 \delta_2 + \sigma_{23} \delta_2 \delta_3 + \sigma_{31} \delta_3 \delta_1 + \sigma_{32} \delta_3 \delta_2 + \sigma_{33} \delta_3 \delta_3 \tag{3}
\]

The "pairs" of basis vectors in equation (3), \(\delta_i \delta_j\), are referred to as \textbf{basis dyads.} Note that each stress component, \(\sigma_{ij}\), has a basis dyad, \(\delta_i \delta_j\), associated with it (\(\sigma_{11}\) gets \(\delta_1 \delta_1\); \(\sigma_{12}\) gets \(\delta_3 \delta_1\); etc.). This is analogous to having each vector component, \(A_i\), associated with a basis vector, \(\delta_i\) (i.e. \(A_1\)
gets $\delta_i$). Note that we are implicitly working in the Cartesian coordinate system since we are using dyads formed from the Cartesian basis vectors $\delta_1, \delta_2,$ and $\delta_3$.

The usefulness in defining the stress tensor $\sigma$ as in equation (3) is that, given any surface (e.g. face of a fluid element) of arbitrary orientation as specified by a unit normal $n$ to that surface, the stress $S$ (force/area) exerted on the surface by the external fluid (the external fluid is the fluid pointed at by $n$) is directly obtained from

$$S = n \cdot \sigma$$

(4)

In equation (4), $n$ is dotted with $\sigma$ from the left; therefore, the dot product is formed by dotting the components of $n$ with the left basis vector of each dyad. For example, for the front face of the element in Fig. 1, $n = \delta_1$. Then (note that the stress components $\sigma_{ij}$ are scalar quantities and can therefore be pulled out of the dot product):

$$S = \delta_1 \cdot \sigma = \sigma_{11} \delta_1 \cdot \delta_1 \delta_1 + \sigma_{12} \delta_1 \cdot \delta_1 \delta_2 + \sigma_{13} \delta_1 \cdot \delta_1 \delta_3 + \sigma_{21} \delta_2 \cdot \delta_1 \delta_1 + \sigma_{22} \delta_2 \cdot \delta_2 \delta_2 + \sigma_{23} \delta_2 \cdot \delta_2 \delta_3 + \sigma_{31} \delta_3 \cdot \delta_1 \delta_1 + \sigma_{32} \delta_3 \cdot \delta_3 \delta_2 + \sigma_{33} \delta_3 \cdot \delta_3 \delta_3$$

$$S = \sigma_{11} \delta_1 + \sigma_{12} \delta_2 + \sigma_{13} \delta_3$$

(5)

Equation (5) is exactly the same as equation (1), if equation (1) is divided by the area $dx_2dx_3$ in order to convert the force $F$ into the stress $S$.


2.i. The Concept of the Control Volume. In transport phenomena it is particularly convenient to apply the fundamental balance laws (i.e. conservations of mass, conservation of momentum, and conservation of energy) to a fixed control volume through which a fluid is flowing (Fig. 2). A fixed control volume is simply a region of space that does not change position or shape or move in any way with respect to the chosen reference frame. Physically, the control volume could be the interior of a chemical reactor, a section of a blood vessel or a pipe, a lake, the atmosphere, etc… We will now apply the law of energy conservation to a fixed control volume.

In what follows, we use $V'$ to denote the region of space corresponding to the control volume, and $B$ to denote the surface that defines the boundary between the interior and exterior of $V'$. 
2.ii. Derivation of the Integral Law of Energy Conservation. Let's say we have a fixed control volume $V'$ of arbitrary size and shape. Then the law of energy conservation can be written:

\[
\frac{\text{rate of accumulation of energy in } V'}{\text{energy in } V'} = \frac{\text{rate of energy transfer into } V'}{\text{by fluid flowing}} + \frac{\text{rate of energy transfer into } V'}{\text{by heat and work}} \tag{6}
\]

Energy is transferred by convection (1st term on right) because the fluid that enters $V'$ brings energy with it. Due to its nonzero velocity, the fluid possesses kinetic energy. Due to its position in a gravitational field (and / or other potential fields, such as electric or magnetic), the fluid possesses potential energy. Due to the molecular bond vibration, rotation, translation, etc. of the fluid molecules, the fluid possesses internal energy. The fluid that enters $V'$ brings the kinetic, potential, and internal energy it has with it, and this is the source of the convective term in equation (6). The second term on the right of equation (4) is present because transfer of heat into $V'$ and the performance of work by the external fluid on the fluid inside $V'$ also result in a transfer of energy. For example, even in the absence of convection, when first term on right equals zero, heat can still flow into $V'$ by conduction or radiation.

Work can be performed on the fluid in $V'$ by various means. Some type of machinery, such as a rotating impeller, could exert force on the fluid inside $V'$, resulting in a displacement of the fluid. The product of this force times the displacement leads to shaft work. Another example of work performed on the fluid inside a control volume is flow work, which represents work done by stresses as they push fluid through the surface of $V'$. We will say more about these types of work later.

We will denote the amount of energy per volume as $\rho e$. $e$ is energy per unit mass of fluid (specific energy), given by

\[
e = u + \frac{1}{2}v^2 + gz \tag{7}
\]

In equation (7), $u$ is internal energy per unit mass of fluid, $(1/2)v^2$ is kinetic energy per unit mass of fluid, and $gz$ is gravitational potential energy per unit mass of fluid (we will assume that the only type of potential energy present is gravitational). Equation (7) is obtained by dividing the total energy $E$ of a mass $m$ of fluid that is moving with speed $v$ and is at a height $z$ by its mass $m$,
\[ e = E/m = (1/m)(U + (1/2)mv^2 + mgz) = u + (1/2)v^2 + gz \]  

(8)

where \( U \) is the total internal energy of the mass \( m \) of fluid, \((1/2)mv^2\) is its kinetic energy, and \( mgz \) is its potential energy due to gravity. Multiplication of \( e \), energy per mass, by mass per volume \( \rho \), yields energy per volume \( \rho e \).

An infinitesimal volume element \( dV' \) inside the control volume \( V' \) contains an amount of energy equal to \( \rho e \, dV' \) (we are just multiplying energy per volume, \( \rho e \), by the volume considered, \( dV' \)). In general \( \rho \) and \( e \) are both functions of position. The total amount of energy \( E \) in the entire control volume \( V' \) is then obtained by integration (i.e. summation) over \( V' \),

\[
E = \iiint_{V'} \rho e \, dV'
\]  

(9)

The derivative of this integral with respect to time is the rate at which the amount of energy in the control volume \( V' \) changes, and therefore represents the rate of accumulation of energy in \( V' \),

\[
\frac{dE}{dt} = \frac{d}{dt} \iiint_{V'} \rho e \, dV'
\]  

(10)

Equation (10) is the desired mathematical expression for the left hand accumulation term in the law of energy conservation stated in equation (6).

Next, an expression for the convective term (1st term on right of equation (6)) will be derived. The rate at which energy is carried by fluid flow across a differential area \( dB \) on the surface of \( V' \) is equal to \(- \rho e (v \cdot n) \, dB\), where \( v \) is the fluid velocity and \( n \) is a unit normal to the surface. Let's dissect this expression to clarify its meaning. \(- v \cdot n\) is the projection of \( v \) onto the normal direction (the \( n \) direction) to the surface, and equals the speed of the fluid perpendicular to the surface (Fig. 3); the minus sign ensures that the speed is positive when fluid flows from outside of \( V' \) into \( V' \). It is customary to regard fluid inflow as positive since it contributes to accumulation in the control volume, while fluid outflow is regarded as negative.
When \(- \mathbf{v} \cdot \mathbf{n}\) is multiplied by \(dB\) the result, \(- \mathbf{v} \cdot \mathbf{n} dB\), equals the rate at which fluid volume flows across \(dB\) (Fig. 4); that is, \(- \mathbf{v} \cdot \mathbf{n} dB\) is the volumetric flowrate into \(V'\) through the area \(dB\) (in units of volume/time). Multiplication of the volumetric flowrate \(- \mathbf{v} \cdot \mathbf{n} dB\) by the energy per volume of fluid \(\rho e\) gives the convective inflow of energy into \(V'\) through \(dB\). Thus \(- \rho e \mathbf{v} \cdot \mathbf{n} dB\) is the volumetric flowrate into \(V'\) through the area \(dB\) (in units of volume/time). Multiplication of the volumetric flowrate \(- \mathbf{v} \cdot \mathbf{n} dB\) by the energy per volume of fluid \(\rho e\) gives the convective inflow of energy into \(V'\) through \(dB\). To arrive at the total convective transport of energy into \(V'\), all of the differential contributions \(- \rho e \mathbf{v} \cdot \mathbf{n} dB\) must be summed (i.e. integrated) over the entire surface \(B\)

\[
\text{Rate of convection of energy into } V' = \iiint_B \rho e (\mathbf{v} \cdot \mathbf{n}) dB
\]

Equation (11) is the desired result for the convection term (1st term on right) in the energy conservation law stated in equation (6).

We still need to deal with the last term in equation (6), the rate of energy transfer into \(V'\) by heat and work. For now we will simply write the total rate of heat transfer into \(V'\) as \(dQ/dt\), and the total rate at which the fluid in the control volume expends energy by doing work on its surroundings as \(dW/dt\). Then, combining the heat and work terms with equations (10) and (11) according to the energy conservation law equation (6) results in

\[
\frac{d}{dt} \left( \iiint_{V'} \rho e dV' \right) = \iiint_B \rho e (\mathbf{v} \cdot \mathbf{n}) dB + \frac{dQ}{dt} - \frac{dW}{dt}
\]

Equation (12) is the integral (control volume) equation of energy conservation. The term on the left hand side is the rate of accumulation of energy in the control volume and the first term on the right is the rate at which energy is brought into the control volume by convection. The 2nd term on the right is the rate at which heat is transferred into the control volume by processes other than convection. The 3rd term on the right, \(- dW/dt\), is the rate at which the surroundings perform work on the fluid in the control volume. Accordingly, the negative of this term (that is, \(dW/dt\) without the minus sign in front) is the rate at which the fluid in \(V'\) performs work on the surroundings. Equation (12) simply states that the rate at which energy is accumulated in \(V'\) equals the rate at which energy is brought into \(V'\) by the flowing fluid, plus the rate at which heat...
is added to \( V' \) by non-convective processes, plus the rate at which the surroundings perform work on the material contained in \( V' \).

Equation (12) will be used to derive a differential form of the energy conservation law. Before we do that, however, let's consider the rate of heat transfer term \( dQ/dt \) and the rate of work term \( dW/dt \) in more detail. Heat can be added to \( V' \) by conduction of heat across the surface of the control volume into its interior. Heat can also be \textit{generated} inside \( V' \) by some "externally coupled mechanism." A common example of such a mechanism is resistive heating of the interior of a wire carrying an electrical current, in which electrical energy provided by an outside source is converted to heat within the wire due to its resistance to current flow. This mechanism of heat generation is "externally-coupled" in the sense that it is not provided by an energy source originating from inside \( V' \); instead the original energy used to generate the heat comes from an external source (i.e. an electrical power plant) that is located outside of \( V' \). If the mechanism were not externally-coupled, then we would be simply converting one type of energy inside \( V' \) into another type of energy inside \( V' \), and the net change in total energy inside \( V' \) would be zero.

Transfer of heat into \( V' \) by conduction can be calculated from the conductive heat flux \( q_F \) (see also Handout 2) as follows. At each point on the surface \( B \) bounding \( V' \), we calculate the component of \( q_F \) perpendicular to the surface as \(- n \cdot q_F \). This perpendicular component of \( q_F \) contributes to heat flux across \( B \), and is expressed in units of energy / (area time). Then \(- n \cdot q_F dB \) equals the rate of heat flow across a differential area \( dB \) (\(- n \cdot q_F dB \) has units of energy/time). Integration over the entire surface \( B \) then sums the contributions from all area elements \( dB \), resulting in the total rate of heat flow through \( B \) by conduction:

\[
\text{Rate of heat flow into } V' \text{ by conduction} = - \int_B n \cdot q_F dB \quad \text{(energy /time)} \quad (13)
\]

In addition to heat conducted across the surface \( B \), heat can also be added directly to \( V' \) via some externally-coupled mechanism, such as conversion of electrical to heat energy as discussed above. Such heat is typically thought of as being generated inside \( V' \) at a volumetric rate \( \dot{q} \), with \( \dot{q} \) possessing units of energy / (volume time). Then, the rate of heat generation (energy/time) inside a volume \( dV' \) is \( \dot{q} dV' \). The total rate of all such volumetric rates of heat generation, summed over the entire volume \( V' \), is

\[
\text{Rate of heat generation in } V' \text{ by externally-coupled mechanisms} = \iiint_{V'} \dot{q} dV' \quad (14)
\]

Replacement of the \( dQ/dt \) term in (12) with the sum of equations (13) and (14) leads to

\[
\frac{d}{dt} \left[ \iiint_{V'} \rho e dV' \right] = - \iiint_B \rho e (v \cdot n) dB - \iiint_B n \cdot q_F dB + \iiint_{V'} \dot{q} dV' - \frac{dW}{dt} \quad (15)
\]

A few remarks are now in order regarding the rate \( dW/dt \) at which the system does work on the surroundings. \( dW/dt \) can be decomposed into two contributions. The first contribution represents work due to forces between the fluid in \( V' \) and the external environment; these forces act at all
points on the surface $B$ where the internal fluid (i.e., fluid inside $V'$) comes in contact and "pushes on" and moves the material outside $V$. This contribution represents the rate of flow work. The second contribution comes from performance of shaft work, and represents the rate at which moving machinery, which is part of the surroundings, does work on the fluid inside $V'$. The classical example of shaft work is that of a rotating impeller that agitates the fluid; since the impeller exerts a force on the fluid that results in a displacement of the fluid through some distance, this force times displacement results in work being done.

The overall work rate term, due to flow and shaft work, comes from performance of work by forces acting between material inside and outside the control volume, and can be constructed as follows. From equation (4) we know that the stress $S$ (force per area), exerted by material outside $V'$ on material inside $V$, is given by $S = n \cdot \sigma$ (recall that $\sigma$ is the stress tensor as defined in equation (3)). Then $- n \cdot \sigma$ is the stress exerted by material in the system on that in the surroundings (note the minus sign). Multiplying $- S$ by $dB$ results in $- n \cdot \sigma dB$ which is the force exerted by the material in $V'$ on the surroundings across a differential surface area $dB$ at the imaginary boundary between the internal and external environments. Dotting this force with the rate of displacement (i.e. the velocity) produces $- (n \cdot \sigma) \cdot v dB$, the rate at which work is performed by the system on the surroundings across the area $dB$. Integrating over the entire surface $B$ is then the total rate of work performed by forces acting between material inside and outside of $V'$,

$$\text{Rate of work at control volume surface} = - \iiint_B n \cdot \sigma \cdot v dB \quad (16)$$

In this derivation, note that we are including both flow and shaft work in equation (16). The shaft work comes from work done at fluid-machinery boundaries; there is no fluid flow across these regions of $B$ where the machinery, which is part of the external surroundings, comes into contact with the material inside $V'$. The flow work comes from forces acting at regions of $B$ across which there is fluid flow. Inserting equation (16) into (15) results in,

$$\frac{d}{dt} \left( \iiint_{V'} \rho e dV' \right) = - \iiint_{B} \rho e (v \cdot n) dB - \iiint_{B} n \cdot q, dB + \iiint_{V'} q dB' + \iiint_{B} n \cdot \sigma \cdot v dB \quad (17)$$

Our main interest in equation (17) will be to use it in derivation of a differential equation that states the principle of energy conservation. We will then be able to use the differential equation to calculate energy-related quantities, such as temperature, inside a flowing fluid.

We first recall two useful definitions: (i) Gauss's Divergence Theorem and (ii) the substantial derivative, also called the material derivative.

Gauss's Divergence Theorem. The Divergence Theorem is used to convert flux integrals of vectors over the control volume surface $B$ to integrals over the control volume interior:
\[ \iiint_n \mathbf{n} \cdot \mathbf{Gd}B = \iiint_{V'} \nabla \cdot \mathbf{Gd}V' \]  

(18)

As before, \( V' \) is a fixed control volume and \( B \) is a closed surface that encloses the control volume. \( n \) is a unit normal vector to \( B \), and \( \mathbf{G} \) is an arbitrary vector whose derivatives are defined throughout \( V' \).

**Substantial (Material) Derivative.** The "material derivative" \( \frac{D}{Dt} \) is defined as

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \]  

(19)

For example, if we apply the material derivative to \( T \), the temperature, we obtain

\[ \frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \]  

(20)

What is the physical meaning of equation (20)? The first term on the right represents rate of change due to time dependence of \( T \), while the second term represents rate of change due to moving through a gradient of \( T \) with a velocity \( \mathbf{v} \). Imagine that a stationary observer is located at a point inside a pool of fluid that is being heated (Fig. 5a). Then, at the fixed location at which the observer is located (the velocity \( \mathbf{v} \) of the observer is zero), the observer would see the temperature rise as the fluid becomes warmer. In this situation, the temperature of the fluid is not at steady state; i.e. it changes with time. At the observer's fixed position, the rate of rise in temperature corresponds to the “unsteady state term” \( \frac{\partial T}{\partial t} \) in equation (20).

![Fig. 5](image)

Now, imagine that an observer is in a pool of fluid in which the temperature is at steady state (i.e. not changing with time) so that, at each point in the fluid, \( \frac{\partial T}{\partial t} = 0 \). However, imagine also that a gradient in the temperature \( \nabla T \) is maintained from one end of the pool to the other (i.e. the fluid is hotter at one end of the pool but cooler at the other; Fig. 5b). If the observer
remains stationary, as time goes on the observer would experience the same temperature since \( \partial T / \partial t = 0 \). However, if the observer begins to move from the cool end of the pool to the warm end, the observer would experience a rate of rise in temperature given by the rate of change of temperature with position (this rate is the gradient \( \nabla T \)) times the speed with which the observer is moving through this gradient from the cooler to the warmer end. This product of speed times the temperature gradient is the \( \mathbf{v} \cdot \nabla T \) term in equation (20), where \( \mathbf{v} \) is the observer’s velocity. Let’s call this term the “motion-through-a-gradient” effect. The dot product ensures that only that component of \( \mathbf{v} \) that results in motion in the direction of \( \nabla T \) contributes to the observed rate of temperature increase. For example, if the observer were not moving up the gradient \( \nabla T \) but only across it, meaning that \( \mathbf{v} \) is perpendicular to \( \nabla T \), then no change in temperature would be experienced since the observer would move along a curve along which the temperature is constant; i.e. along an isotherm.

Now imagine that the observer is instead an element of fluid moving with the flow velocity \( \mathbf{v} \). Therefore, \( \mathbf{v} \) is now the same velocity that appears in the Navier-Stokes equations, the continuity equation, etc. Then, for the general situation in which (i) the fluid is being heated or cooled, so that the temperature is not at steady state and (ii) temperature gradients exist from point to point in the fluid, the total rate of change of temperature \( DT/\partial t \) that the fluid element will experience is the sum of the unsteady state and the motion-through-a-gradient terms,

\[
\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T
\]

This is the material derivative, as defined in (20). The material derivative represents the rate of change of a property (temperature in the above example) experienced by a fluid element as it moves with the flow velocity \( \mathbf{v} \).

**Differential Law of Energy Conservation.** Recall the integral equation of energy conservation

\[
\frac{d}{dt} \left[ \iiint_V \rho c_v \mathbf{V}' \cdot \, dV' \right] = - \iiint_B \mathbf{n} \cdot \rho c_v \mathbf{v} \cdot dB + \iiint_B \mathbf{n} \cdot \mathbf{q} \cdot dB + \iiint_V \dot{q} \, dV' + \iiint_B \mathbf{n} \cdot \mathbf{\sigma} \cdot \mathbf{v} \cdot dB \tag{17}
\]

This equation states: accumulation of internal+kinetic+potential energy in a control volume \( V \) (lhs) equals convection of internal+kinetic+potential energy into \( V \) (1st term on right), plus conduction of heat into \( V \) (second term on right), plus generation of heat inside \( V \) by externally-coupled mechanisms, plus work performed on material in \( V \) by forces acting at the boundary between \( V' \) and surroundings. Converting all surface integrals into volume integrals using the Divergence Theorem (18), and moving the operator \( d/dt \) in the accumulation term inside the integral yields (note: we can move the \( d/dt \) inside the accumulation integral because \( V' \) is a fixed control volume, so that the limits of integration over \( V' \) are not functions of time),

\[
\iiint_V \frac{\partial (\rho c_v \mathbf{V}' \cdot \mathbf{v})}{\partial t} \, dV' = - \iiint_V \nabla \cdot (\rho c_v \mathbf{V}' \cdot \mathbf{V}) - \iiint_V \nabla \cdot \mathbf{q} \, dV' + \iiint_V \dot{q} \, dV' + \iiint_V \nabla \cdot (\mathbf{\sigma} \cdot \mathbf{v}) \, dV' \tag{21}
\]
In the fourth term on the right in equation (17), note that $\sigma \cdot v$ is a vector obtained by taking $\sigma$ as defined in equation (3) and dotting it with the velocity vector $v$ from the right - in this operation, the right basis vector of each dyad in $\sigma$ is dotted with the basis vectors in $v$. The main point is that, since $\sigma \cdot v$ is just a vector, the divergence theorem is used here in the same way as for the other terms on the right even though the notation may look more complicated. Combining all terms under a single integration sign,

$$\iiint_V \left( \frac{\partial (\rho e)}{\partial t} + \nabla \cdot \rho ev + \nabla \cdot q_F - \dot{q} - \nabla \cdot (\sigma \cdot v) \right) dV' = 0$$

(22)

The volume $V'$ over which the integration is performed is arbitrary in that we are not allowed to select a particular $V'$ (i.e. by defining the limits of integration) so as to satisfy the equality to zero in (22). Rather, equation (22) must hold for any control volume (as long as it’s a fixed control volume). In such a case, the only way to satisfy the equality to zero is by requiring the integrand to be zero,

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot \rho ev + \nabla \cdot q_F - \dot{q} - \nabla \cdot (\sigma \cdot v) = 0$$

(23)

Equation (23) is the **differential law of energy conservation**. It could have been obtained just as well by performing an energy balance on a differential volume element, such as a differential cube. Instead, we performed an energy balance on a macroscopic control volume $V'$, and then showed that (23) must hold at each point in space. What are the physical meanings of the terms in (23)? These can be traced back to the original integral conservation law. For example, the first term in equation (23) came from the accumulation term in the integral equation (17); therefore, this term must represent the rate of accumulation per volume of total (internal + kinetic + potential) energy. Since (23) is a differential equation, this term applies to a point in space rather than to some large control volume. Similarly, $- \nabla \cdot \rho ev$ can be traced to be the rate at which energy is convected to that point, $- \nabla \cdot q_F$ the rate at which heat is conducted to that point, $\dot{q}$ the rate at which energy is generated at that point by externally-coupled processes, and $\nabla \cdot (\sigma \cdot v)$ the rate at which surroundings perform work on the fluid at that point.

We will now rearrange the differential energy balance, equation (23). First, using the product rule of differentiation we recognize that $\partial p e/\partial t = \rho \partial e/\partial t + e \partial \rho /\partial t$. Second, we could show that $\nabla \cdot \rho ev = e \nabla \cdot \rho v + \rho v \cdot \nabla e$ (e.g. see identity 30d in handout 1). With these changes, (23) can be written

$$\rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t} + e \nabla \cdot \rho v + \rho v \cdot \nabla e + \nabla \cdot q_F - \dot{q} - \nabla \cdot (\sigma \cdot v) = 0$$

(24)

We recall the **equation of continuity** (the differential law of mass conservation) from fluid mechanics,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = 0$$

(25)
According to (25), the sum of the 2nd and 3rd terms in (24) is zero; furthermore, the sum of the 1st plus 4th terms can be written as \( \rho (\partial e/\partial t + \mathbf{v} \cdot \nabla e) = \rho \mathbf{De}/\partial t \) where we used the material derivative notation. The differential law of energy conservation can then be rewritten as

\[
\rho \frac{\mathbf{De}}{\partial t} = -\nabla \cdot \mathbf{q} + \dot{q} + \nabla \cdot (\mathbf{\sigma} \cdot \mathbf{v})
\]  

(26)

Equation (26) states that the rate of change of the total energy (internal + kinetic + potential) of a fluid element moving with the flow (left hand side; note that a fluid element moving with the flow is a system with zero net convection of matter in or out) is equal to the rate of heat flow into the element by conduction (1st term on right), plus the rate of heat generation within the fluid element from externally coupled sources (2nd term on the right), plus the rate at which work is done on the fluid element by the surrounding fluid (last term on the right). The last term can be further broken down into useful work and work that is dissipated to internal energy, as discussed later. For now, we recognize that this last term looks somewhat confusing, so to clarify how it would be evaluated in practice let's work it out in detail. Taking the definition of \( \mathbf{\sigma} \) from equation (3), we have

\[
\nabla \cdot (\mathbf{\sigma} \cdot \mathbf{v}) = \nabla \cdot \{ (\sigma_{11} \delta_{11} + \sigma_{12} \delta_{12} + \sigma_{13} \delta_{13} + \sigma_{21} \delta_{21} + \sigma_{22} \delta_{22} + \sigma_{23} \delta_{23} + \sigma_{31} \delta_{31} + \sigma_{32} \delta_{32} + \sigma_{33} \delta_{33}) \cdot (v_{11} \delta_{11} + v_{22} \delta_{22} + v_{33} \delta_{33}) \}
\]

Carrying out the dot product between \( \mathbf{\sigma} \) and \( \mathbf{v} \), in which the right vector of each dyad is dotted into the three components of \( \mathbf{v} \) (note we are using the right dyad vector because \( \mathbf{v} \) is dotted into \( \mathbf{\sigma} \) from the right) results in

\[
\nabla \cdot (\mathbf{\sigma} \cdot \mathbf{v}) = \nabla \cdot \{ \sigma_{11} v_{11} \delta_{11} + \sigma_{12} v_{12} \delta_{12} + \sigma_{13} v_{13} \delta_{13} + \sigma_{21} v_{21} \delta_{21} + \sigma_{22} v_{22} \delta_{22} + \sigma_{23} v_{23} \delta_{23} + \sigma_{31} v_{31} \delta_{31} + \sigma_{32} v_{32} \delta_{32} + \sigma_{33} v_{33} \delta_{33} \}
\]

Inserting the definition of the gradient operator (in Cartesian coordinates)

\[
\nabla \cdot (\mathbf{\sigma} \cdot \mathbf{v}) = (\frac{\partial}{\partial x_1} \delta_{11} + \frac{\partial}{\partial x_2} \delta_{12} + \frac{\partial}{\partial x_3} \delta_{13}) \cdot (\sigma_{11} v_{11} \delta_{11} + \sigma_{12} v_{12} \delta_{12} + \sigma_{13} v_{13} \delta_{13} + \sigma_{21} v_{21} \delta_{21} + \sigma_{22} v_{22} \delta_{22} + \sigma_{23} v_{23} \delta_{23} + \sigma_{31} v_{31} \delta_{31} + \sigma_{32} v_{32} \delta_{32} + \sigma_{33} v_{33} \delta_{33})
\]
\[ \nabla \cdot (\sigma \cdot v) = \frac{\partial (\sigma_{11} v_1 + \sigma_{12} v_2 + \sigma_{13} v_3)}{\partial x_1} + \frac{\partial (\sigma_{21} v_1 + \sigma_{22} v_2 + \sigma_{23} v_3)}{\partial x_2} + \frac{\partial (\sigma_{31} v_1 + \sigma_{32} v_2 + \sigma_{33} v_3)}{\partial x_3} \]  
\text{(27)}

Alternately, using the summation convention, (27) can be written more compactly as

\[ \nabla \cdot (\sigma \cdot v) = \frac{\partial (\sigma_{ij} v_j)}{\partial x_i} \]  
\text{(28)}

(can you reverse expand (28) into (27)?) If we knew the dependence of all the stress and velocity components (\(\sigma_{ij}\)'s and \(v_i\)'s) on the coordinate variables \(x_i\) we could directly evaluate (28) and arrive at a formula for the rate at which surroundings do work on the fluid at a point. There’s not much in the above manipulations in terms of difficulty, but they can be confusing when seen for the first time and can become large in terms of paper consumed (hence the frequent use of the summation convention). Note that (28) is written for the Cartesian coordinate system.

Conservation of energy is a physical law; therefore equation (26) is general. It can be specialized to Newtonian fluids by inserting in the Newtonian fluid model for the stress \(\sigma\) (equation (2)), and to fluids obeying Fourier's Law by substituting that law for the conductive heat flux \(q_F\). We will make these substitutions after rearranging (26) into a more commonly used form.

### 3. Related Energy Equations.
#### 3.i. Mechanical Energy Balance.

The Mechanical Energy Equation is concerned with kinetic and potential energies only. The internal energy is not included. Therefore, this equation will not represent the law of energy conservation, since it is not a total energy balance. The Mechanical Energy Equation can be derived directly from the rate of change of momentum equation studied in fluid mechanics. The momentum equation of interest is

\[ \rho \frac{Dv_1}{Dt} \delta_1 + \rho \frac{Dv_2}{Dt} \delta_2 + \rho \frac{Dv_3}{Dt} \delta_3 = \rho g \delta_1 + \rho g_2 \delta_2 + \rho g_3 \delta_3 + \\
\left( \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \right) \delta_1 + \left( \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} \right) \delta_2 + \left( \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \right) \delta_3 \]  
\text{(29)}

Here we wrote the momentum balance (29) as one vector equation, rather than as three scalar equations (one for each coordinate direction). By combining the vector components, we could equivalently have also written equation (29) as

\[ \rho \frac{Dv}{Dt} = \rho g + \nabla \cdot \sigma \]  
\text{(30)}
where \( \mathbf{v} = v_1 \mathbf{\delta}_1 + v_2 \mathbf{\delta}_2 + v_3 \mathbf{\delta}_3 \), \( \mathbf{g} = g_1 \mathbf{\delta}_1 + g_2 \mathbf{\delta}_2 + g_3 \mathbf{\delta}_3 \), and \( \mathbf{\sigma} \) is given in equation (3). You could verify that the last term \( \nabla \mathbf{\sigma} \) in (30) expands into the corresponding terms in (29) by performing similar manipulations as those used to derive (27). Taking the dot product of equation (30) with the velocity \( \mathbf{v} \) results in

\[
\rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = \mathbf{v} \cdot (\rho \mathbf{g} + \nabla \cdot \mathbf{\sigma})
\]

We could prove that

\[
\rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = \frac{\rho}{2} \frac{D(v \cdot \mathbf{v})}{Dt} = \rho \frac{D(v^2/2)}{Dt}
\]

Using (32), equation (31) becomes

\[
\rho \frac{D(v^2/2)}{Dt} = \rho \mathbf{v} \cdot \mathbf{g} + \mathbf{v} \cdot (\nabla \cdot \mathbf{\sigma})
\]

Now we make use of some facts: (i) \( gz \), the gravitational potential per unit mass at a point at height \( z \), does not change with time so that \( \partial (gz)/\partial t = 0 \), and (ii) if the gravitational acceleration \( g \) is constant, then the gravitational force per unit mass \( \mathbf{g} = -g \mathbf{\delta}_z = -g \nabla z = -\nabla (gz) \). Here, \( \mathbf{\delta}_z = \nabla z \) is a unit vector pointing in the positive \( z \) direction. Adding \( \rho \partial (gz)/\partial t \) to the left hand side and substituting \( -\nabla (gz) \) for \( \mathbf{g} \), equation (33) rearranges to

\[
\rho \frac{D(v^2/2)}{Dt} + \rho \left[ \frac{\partial (gz)}{\partial t} + \mathbf{v} \cdot \nabla (gz) \right] = \mathbf{v} \cdot (\nabla \cdot \mathbf{\sigma})
\]

It was alright to add \( \rho \partial (gz)/\partial t \) only to the left hand side since this term equals 0. Using the definition of the material derivative, \( \frac{\partial (gz)}{\partial t} + \mathbf{v} \cdot \nabla (gz) = D(gz)/Dt \), and the above expression rearranges to

\[
\rho \frac{D(v^2/2 + gz)}{Dt} = \mathbf{v} \cdot (\nabla \cdot \mathbf{\sigma})
\]

Equation (34) is called the **Mechanical Energy Balance**. The term "mechanical" emphasizes that the equation is only concerned with macroscopic kinetic and gravitational potential energies and does not include the molecular internal energy. However, it will next be used to help derive a differential energy balance on internal energy.

**Differential Balance on Internal Energy.** A balance on internal energy is obtained by subtracting the balance on mechanical (kinetic + potential) energy, equation (34), from the balance on total (internal + kinetic + potential) energy, equation (26). We have
\[
\begin{align*}
\rho \frac{De}{Dt} &= -\nabla \cdot q_F + \dot{q} + \nabla \cdot (\sigma \cdot v) \quad \text{(total energy balance)} \\
\rho \frac{D(v^2/2 + gz)}{Dt} &= v \cdot (\nabla \cdot \sigma) \quad \text{(mechanical energy balance)}
\end{align*}
\]

Recalling that \( e = u + v^2/2 + gz \), subtraction of the mechanical energy equation from the total energy balance leads to

\[
\rho \frac{D(u + v^2/2 + gz - v^2/2 - gz)}{Dt} = -\nabla \cdot q_F + \dot{q} + \nabla \cdot (\sigma \cdot v) - v \cdot (\nabla \cdot \sigma) \quad (35)
\]
or

\[
\rho \frac{Du}{Dt} = -\nabla \cdot q_F + \dot{q} + \nabla \cdot (\sigma \cdot v) - v \cdot (\nabla \cdot \sigma) \quad (36)
\]

Equation (36) has not referred to any particular coordinate system as yet. However, for the rest of the derivation we will make use of Cartesian coordinates. This is not necessary, but may be easier to follow than staying in the general vector notation. We will need the summation and Kronecker delta conventions, so you may wish to refresh these concepts from Handout 1.

We also recall that the divergence of a vector \( \mathbf{A} \) is

\[
\nabla \cdot \mathbf{A} = \left( \frac{\partial}{\partial x_1} \delta_1 + \frac{\partial}{\partial x_2} \delta_2 + \frac{\partial}{\partial x_3} \delta_3 \right) \cdot (A_1 \delta_1 + A_2 \delta_2 + A_3 \delta_3) = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}
\]

\[
\nabla \cdot \mathbf{A} = \frac{\partial A_i}{\partial x_i} \quad (37)
\]

Second, from equation (28)

\[
\nabla \cdot (\sigma \cdot v) = \frac{\partial (\sigma_{ij} v_j)}{\partial x_i} \quad (28)
\]

Third, by procedures similar to those used to derive equation (28), we could show that

\[
v \cdot (\nabla \cdot \sigma) = v_j \frac{\partial \sigma_{ij}}{\partial x_i} \quad (38)
\]

Using equations (37), (28), and (38), equation (36) rearranges to
\[ \rho \frac{Du}{Dt} = -\frac{\partial q_{Fi}}{\partial x_i} + \dot{\varrho} + \frac{\partial \sigma_{ij} v_j}{\partial x_i} - v_j \frac{\partial \sigma_{ij}}{\partial x_i} \]  

(39)

According to the product rule of differentiation

\[ \frac{\partial \sigma_{ij} v_j}{\partial x_i} = \sigma_{ij} \frac{\partial v_j}{\partial x_i} + v_j \frac{\partial \sigma_{ij}}{\partial x_i}, \]

thus

\[ \rho \frac{Du}{Dt} = -\frac{\partial q_{Fi}}{\partial x_i} + \dot{\varrho} + \sigma_{ij} \frac{\partial v_j}{\partial x_i} \]  

(40)

Using Fourier's Law, \( q_{Fi} = -k \frac{\partial T}{\partial x_i} \), and the Newtonian fluid expression for \( \sigma_{ij} \) from equation (2) (we are now specializing the internal energy balance to Newtonian fluids that obey Fourier's Law):

\[ \rho \frac{Du}{Dt} = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \dot{\varrho} + \left\{ -p \delta_i + \lambda \frac{\partial v_k}{\partial x_k} \delta_j + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} \frac{\partial v_j}{\partial x_i} \]  

(41)

The last term on the right (the term inside the \( [ \ ] \) parentheses) becomes

\[ -p \frac{\partial v_j}{\partial x_i} \delta_i + \lambda \left( \frac{\partial v_k}{\partial x_k} \frac{\partial v_j}{\partial x_i} \right) \delta_j + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial v_j}{\partial x_i} \]

Summing over \( j \) in the two terms involving the Kronecker delta results in (recall that the Kronecker delta \( \delta_{ij} \) equals zero unless \( i = j \)):

\[ -p \frac{\partial v_i}{\partial x_i} + \lambda \frac{\partial v_k}{\partial x_k} \frac{\partial v_i}{\partial x_i} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial v_j}{\partial x_i} \]  

(42)

Using expression (37) for the divergence of a vector, we see that the \( p \) in (42) is multiplied by \( (\nabla \cdot \mathbf{v}) \) and the \( \lambda \) in (42) is multiplied by \( (\nabla \cdot \mathbf{v})^2 \). Replacing the last term in (41) with expression (42) rearranges equation (41) to

\[ \rho \frac{Du}{Dt} = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \dot{\varrho} - (\nabla \cdot \mathbf{v}) + \lambda (\nabla \cdot \mathbf{v})^2 + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial v_j}{\partial x_i} \]  

(43)

The last two terms in (43) are usually combined into the **dissipation function** \( \Phi \),

\[ \Phi = \lambda (\nabla \cdot \mathbf{v})^2 + \mu \left( \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + \frac{\partial v_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \right) \]

(44)

or, summing over the \( i \) and \( j \) indices,
\[ \Phi = \lambda (\nabla \cdot \mathbf{v})^2 + 2\mu \left[ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right]^2 + \mu \left[ \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right)^2 + \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right)^2 + \left( \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \right)^2 \right] \]  

(45)

To get to (45), the dissipation function was first expanded by summing over both \( i \) and \( j \) indices in Equation (44) and then simplified by collecting terms. Using the dissipation function, the internal energy balance (43) becomes,

\[ \rho \frac{Du}{Dt} = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + q - p (\nabla \cdot \mathbf{v}) + \Phi \]  

(46)

Equation (46) states that the rate of change of internal energy of a fluid element moving with the flow (left hand side; recall such an element is a closed system as no material enters or leaves) equals the rate of heat flow into the element by conduction (1st term on the right), plus any externally-coupled rate of heat generation within the fluid element (2nd term on right), plus the rate at which work is performed on the element by pressure forces (3rd term on right), plus the rate at which mechanical energy is converted to internal energy within the element due to frictional dissipation (4th term on right). The physical meaning of the third term on the right can be deduced by showing that \( \nabla \cdot \mathbf{v} \) is directly proportional to the volumetric rate of expansion of a fluid element (we will omit this proof); this interpretation of \( \nabla \cdot \mathbf{v} \) then leads to the conclusion that \(- p (\nabla \cdot \mathbf{v})\) is the rate at which so-called "\( pV \) work" (perhaps similar terminology is familiar from thermodynamics) is performed on the fluid element by using pressure to compress or to expand the material inside it. The rigorous identification of \( \Phi \) as the rate at which mechanical energy is converted to internal energy by viscous dissipation requires appeal to the second law of thermodynamics. We will omit this proof here, but want to recognize that \( \Phi \) represents a very common phenomenon. From experience we know that rubbing two objects together can heat them. This heating comes from dissipation of mechanical energy of the objects’ motion to internal energy. An analogous process occurs in a flowing fluid in which different parts of the fluid that are in contact move at different velocities, leading to such "rubbing" and resulting in a frictional dissipation of mechanical energy to internal energy (why is there no dissipation term in the total energy balance, equation (26)?).

For some materials, such as ideal gases and strictly incompressible liquids and solids, the internal energy \( u \) is only a function of temperature so that \( u = u(T) \). In that case, the **constant volume specific heat capacity** \( c_V \) is defined by

\[ \dot{c}_V = \frac{du}{dT} \]  

(47)

so that
\[ du = \dot{c}_V \, dT \]  \hspace{1cm} (48)

Using equation (48), the rate of change of the internal energy of a fluid element moving with the flow, \( Du/ Dt \), is equal to

\[ Du/ Dt = \dot{c}_V \, DT/ Dt \hspace{1cm} (\text{ideal gases and incompressible liquids}) \]  \hspace{1cm} (49)

where \( DT/ Dt \) is the rate of temperature change of the fluid element. Using equation (49), the internal energy balance equation (46) becomes

\[ \rho \dot{c}_V \frac{DT}{Dt} = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \dot{q} - p \, (\nabla \cdot \mathbf{v}) + \Phi \hspace{1cm} (\text{ideal gases \& incompressible liquids}) \]  \hspace{1cm} (50)

or, if the substantial derivative is expanded,

\[ \rho \dot{c}_V \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \dot{q} - p \, (\nabla \cdot \mathbf{v}) + \Phi \hspace{1cm} (\text{ideal gases \& incompressible liquids}) \]  \hspace{1cm} (51)

After much manipulation, we have arrived at a useful form of the energy balance that will allow us to calculate temperature fields, a quantity of obvious engineering importance.

In this course we will only be concerned with incompressible liquids or solids, or ideal gases, so that equations (50) and (51) apply. However, it is useful to briefly consider also the case when these equations are not good approximations because the internal energy depends on other variables besides temperature. For single component systems (e.g. pure fluids) \( u \) becomes a function of two variables such as \( T \) and \( 1/\rho \). \( 1/\rho \) is volume of fluid per unit mass of fluid, and is customarily referred to as the "specific volume" \( V \). If \( u = u(T, V) \), then

\[ du = (\partial u/\partial T)_V \, dT + (\partial u/\partial V)_T \, dV \]

By definition, the derivative \((\partial u/\partial T)_V = \dot{c}_V \); therefore

\[ du = \dot{c}_V \, dT + (\partial u/\partial V)_T \, dV \]

Because of the extra dependence of \( u \) on \( V \), equations (48) – (51) would pick up another term corresponding to the \((\partial u/\partial V)_T \) \, dV dependence of internal energy on specific volume.

For many common situations equations (50) and (51) simplify into more user-friendly forms. For example, when there is no heat generation in the material by externally-coupled processes, \( \dot{q} = 0 \). Furthermore, if the thermal conductivity \( k \) is assumed constant, then \( k \) can be taken in front of the \( \partial /\partial x_i \) derivative. The remaining \( \partial /\partial x_i \, (\partial T /\partial x_i) = \nabla^2 T \), the Laplacian of \( T \).

With these modifications, equation (51) would read as

\[ \rho \dot{c}_V \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T - p \, (\nabla \cdot \mathbf{v}) + \Phi \hspace{1cm} (k \text{ constant}; \ \dot{q} = 0) \]  \hspace{1cm} (52)

Furthermore, if the flow is incompressible so that \( \rho = \text{constant} \), the \( \nabla \cdot \mathbf{v} \) terms in equations (50) and (51) and in the dissipation function equation (45) disappear. Thus
\[
\rho \hat{c}_V \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + \Phi \quad (\rho, k \text{ constant}; \dot{q} = 0)
\]  

(53)

where \( \Phi \) is given by equation (45) but without the \( \nabla \cdot \mathbf{v} \) term. If, in addition, we are considering the temperature distribution in a solid material or a fluid at rest, then \( \mathbf{v} = 0 \). When \( \mathbf{v} = 0 \), \( \Phi \) also equals zero (see equation (45)), and (53) becomes

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{c}_V} \nabla^2 T \quad (\rho, k \text{ constant}; \mathbf{v} = 0; \dot{q} = 0)
\]  

(54)

For strictly incompressible solids and liquids the density does not depend on temperature, and the constant volume and constant pressure specific heat capacities are equal, \( \hat{c}_V = \hat{c}_P \) (one could prove this from thermodynamic relations and the definitions of \( \hat{c}_V \) and \( \hat{c}_P \)). Since most solids and liquids typically do not expand very much with temperature this equality is often a good approximation, and \( \hat{c}_P \) can be used instead of \( \hat{c}_V \) in the above equations (however, for an ideal gas, a compressible material, \( \hat{c}_P = \hat{c}_V + R \)). If \( \hat{c}_P \) is used instead of \( \hat{c}_V \) in equation (54), the combination of parameters in front of \( \nabla^2 T \) would be \( k/\rho \hat{c}_P \). This particular combination is referred to as the **thermal diffusivity** \( \alpha \),

\[
\alpha = \frac{k}{\rho \hat{c}_P}
\]  

(55a)

and (54) may be written as

\[
\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad (\rho, k \text{ constant}; \mathbf{v} = 0; \dot{q} = 0; \hat{c}_V = \hat{c}_P)
\]  

(55b)

Other simplifications are possible. In any problem the simplifications must reflect the physical nature and statement of the problem.

### 4. Examples of Boundary Conditions.

#### 4.i Conditions on Temperature

A very common boundary condition is one in which temperature is specified at a point in space or an instant in time. For example,

\[
T (x = 0) = T_o \quad (56)
\]

\[
T (t = 0) = T_o \quad (57)
\]

where the value of \( T_o \) is given.

#### 4.ii Conditions on Heat Flux

One example of a boundary condition on heat flux deals with heat conduction at an **adiabatic** boundary or wall. By definition, an adiabatic wall does not transmit heat. From Fourier’s Law, the conductive heat flux \( q_{\perp} \) normal to the boundary is given by (the subscript "\( \perp \)" indicates the component of heat flux normal to the boundary)
if \( z \) is taken to be the coordinate perpendicular to the boundary, as in Fig. 6. Since the boundary is adiabatic, there can be no conductive heat flux perpendicular to it; therefore, \( q_{F\perp} \) must equal zero at the boundary, i.e. at \( z = 0 \). This condition implies that

\[
q_{F\perp} = -k \nabla T = -k \frac{dT}{dz}
\]

(58)

The subscript \( z = 0 \) on the temperature derivative indicates that the derivative is to be evaluated at the boundary, at \( z = 0 \).

Now imagine a heat-permeable (diathermal) boundary between systems I and II, as in Fig. 7, and that steady state applies. Under steady state, there can be no accumulation of heat at the boundary. If heat conduction is the only means by which heat arrives and leaves the boundary, then the rate at which heat is conducted to the boundary, say from system I, must equal the rate at which it is conducted away from the boundary into system II. This means that

\[
k_I \left( \frac{dT}{dz} \right)_{I, z=0} = k_{II} \left( \frac{dT}{dz} \right)_{II, z=0}
\]

(60)

where \( k_I \) and \( k_{II} \) are heat conductivities in systems I and II, respectively, and \( \left( \frac{dT}{dz} \right)_{I, z=0} \) and \( \left( \frac{dT}{dz} \right)_{II, z=0} \) are the temperature gradients in the two systems, evaluated at the boundary.
A third example we consider is that of a heterogeneous reaction occurring at a boundary between a system and an adiabatic wall, Fig. 8. Under steady state conditions there can be no accumulation of heat at the boundary; therefore, the rate at which heat of reaction is generated at the boundary must equal the rate at which it leaves. Since the wall is adiabatic, no heat can flow into the wall; thus, heat can only flow into the system along the direction of increasing \( z \). If conduction is the only mechanism of heat transfer away from the surface, then

\[
\sum_{i=1}^{n} \dot{H}_i r_i^S = k \left( \frac{dT}{dz} \right)_{z=0}
\]  

where \( r_i^S \) is the rate at which mass of species \( i \) is produced by chemical reactions at the boundary in units of mass / (area time), \( n \) is the number of species participating in the reactions, and \( \dot{H}_i \) is the specific enthalpy (units: energy / mass) of species \( i \). The left hand side of equation (61) is the heat generated per area per time by chemical reactions, and the right hand side is the rate at which heat flows away from the surface by conduction. Both terms have units of energy / (area time).

![Fig. 8](image)

The above examples have involved conduction of heat to or from a boundary as the dominant mechanism of heat transfer. Since conduction was involved, Fourier's Law was used to calculate the heat flux. However, in all these examples the heat flux to or from a boundary could also have been modeled using the heat transfer coefficient approach,

\[
q_F = h (T_S - T_\infty)
\]  

with \( T_S \) the temperature at the boundary and \( T_\infty \) the temperature in the bulk of the phase from which the heat occurs. In the case of radiative heat transfer, expressions introduced in the prior handout could be used to calculate radiative heat flux at a boundary.