Examples of Laminar Flows

In laminar flows the fluid moves in "layers" or laminae, in contrast to the apparently chaotic motion of turbulent flow. Laminar flows in many different geometries have been investigated with the help of transport models; here we consider just a few well known examples. We also take this opportunity to introduce the use of **integral equation** models and of the method of **combination of variables** to solve transport problems. In general, when setting up a problem you will need to address several questions:

The 1st question to ask is: What are we trying to find out about the flow (i.e. velocities, pressure, etc)? In other words, what are the unknowns that we are after?

The 2nd question to ask is: What equations do we use? Recall that we need as many equations as we have unknowns, together with any boundary conditions and possibly also relations between intensive thermodynamic variables (such as equations of state).

The 3rd step is to solve the equations for the unknowns (i.e. velocities, pressure) of interest. This can be a long process, and often requires that we make certain approximations along the way, in addition to a fair amount of mathematical manipulations.

Internal Flows

Internal flows are those that occur within a channel or pipe. The fluid body is of finite dimensions and is confined by the channel or pipe walls. At the entry region to a channel, the fluid develops a boundary layer next to the channel walls, while the central "core" of the fluid may remain as a uniform flow (Figure 1). Within the boundary layer, viscous stresses are very prominent, slowing down the fluid due to its friction with the channel walls. This slowdown propagates away from the walls: as the fluid enters the channel the fluid particles immediately next to the walls are slowed down, these particles then viscously interact with and slow down those in the second layer from the wall, and so on. Downstream, the boundary layers therefore thicken and eventually come together, eliminating the central core. Eventually, the velocity assumes some average profile across the channel which is no longer influenced by any edge effects arising from the entrance region. At this point, the flow no longer depends on what has occurred at the channel entrance, and we could solve for its properties (such as the velocity profile) without including an entrance region in the calculations. At this stage, we say that the flow has become "fully developed."

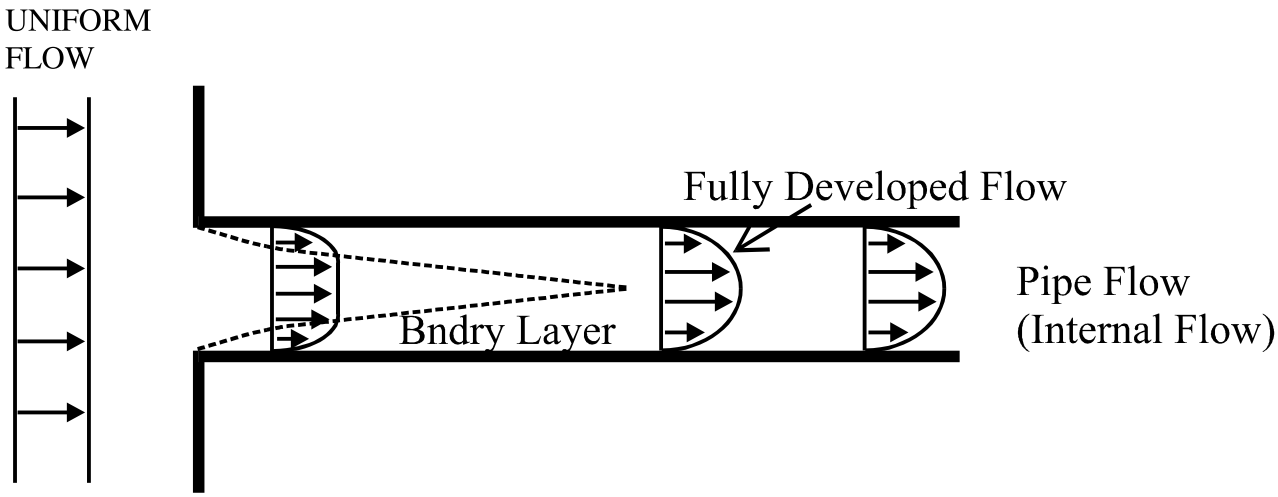

Figure 1.

For flow in a pipe, Boussinesq has estimated that the distance X_L needed for the flow to become fully developed can be approximated by

$$X_L = 0.03ReD \quad (1)$$

where

$$Re = \rho VD/\mu \quad (2)$$

Re is the "Reynold's number," D is the pipe diameter, ρ is the fluid density, μ is the fluid viscosity, and V is the average velocity of the fluid in the pipe (V = volumetric flowrate / cross-sectional area of pipe). When Re is less than about 2300, the flow in a pipe will in general be laminar. In laminar flow, the values of velocity, pressure, and other quantities at a point in space do not fluctuate randomly with time. In laminar flow, fluid flows smoothly. Above $Re \approx 2300$ the flow will usually become turbulent, and thus be characterized by sudden fluctuations in the values of the velocity components, pressure and other variables characterizing the flow. In turbulent flow, the motion of fluid is chaotic and apparently unpredictable. Later we will consider how to apply the differential conservation laws when turbulence is present. For now, we will focus on describing laminar, fully developed flows.

1). Steady-state, laminar flow between stationary, parallel plates. This simple example of an internal, laminar flow is illustrated in Figure 2. We want to develop an expression for the fluid velocity profile v_1 (what independent variables does v_1 depend on?) From the Cartesian version of the differential equation of motion for a Newtonian fluid with constant ρ and μ (equations 14 in Handout 8),

$$0 = - \mathrm{d}p/\mathrm{d}x_1 + \mu(\mathrm{d}^2v_1/\mathrm{d}x_2^2) \quad (3)$$

Note that body forces were assumed to act perpendicular to the direction of flow, and so do not appear in equation (3). Also, the pressure gradient $\mathrm{d}p/\mathrm{d}x_1$ is taken to have a constant value. Applying the no-slip boundary conditions $v_1 = 0$ at $x_2 = h$ and at $x_2 = -h$, (or applying one of these conditions in conjunction with the condition that $\mathrm{d}v_1/\mathrm{d}x_2 = 0$ at the midpoint of the gap between the plates) we find

$$v_1 = 1/(2\mu)\ \mathrm{d}p/\mathrm{d}x_1\ (x_2^2 - h^2) \quad (4)$$

Therefore, the velocity profile is parabolic as depicted in Figure 2. Note that the flow is entirely driven by the presence of a pressure gradient. If $dp/dx_1 = 0$, then $v_1 = 0$ and there is no flow. For a more viscous fluid, a greater pressure gradient is required to achieve a given rate of flow.

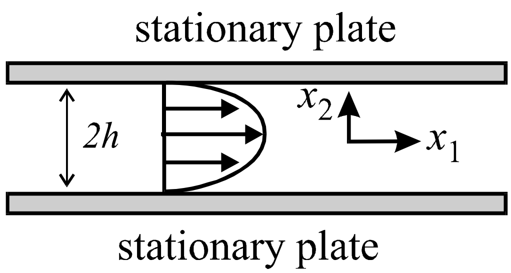

Figure 2

2). Steady-state, laminar flow between two parallel surfaces in relative motion (Couette Flow). When one of the surfaces in Figure 2 is in motion, the flow is often termed "Couette Flow." The solution for the velocity profile follows the same protocol as for example 1, except that one of the surfaces moves at a speed V_1 relative to the other (Figure 3). In general, a nonzero pressure gradient in the direction of flow may be present (the pressure gradient is again assumed to be constant). If body forces do not contribute to the flow, the Cartesian differential equation of motion for a Newtonian fluid with constant ρ and μ may be written as,

$$0 = - dp/dx_1 + \mu(d^2v_1/dx_2^2)$$

which is same as equation (3). However, now the boundary conditions are $v_1 = V_1$ at $x_2 = h$ and $v_1 = 0$ at $x_2 = 0$. The resultant velocity profile is easily found to be

$$v_1 = 1/(2\mu)\, dp/dx_1\, (x_2^2 - hx_2) + V_1x_2 / h \qquad (5)$$

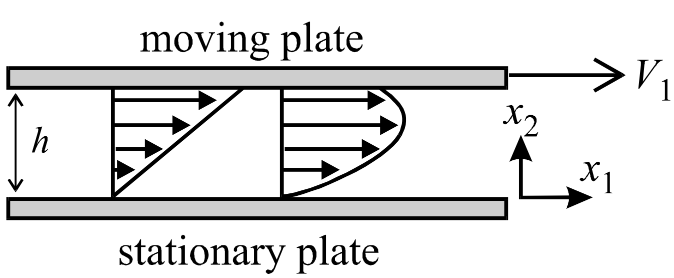

Figure 3. The velocity profile on the left is observed when the pressure gradient in the gap is zero. The velocity profile on the right corresponds to a decreasing pressure in the direction of flow, i.e. $dp/dx_1 < 0$.

Equation 5 shows that the velocity profile is a combination of parabolic (1st term on the right) and linear (2nd term on the right) flows. Note that the parabolic contribution is entirely driven by the presence of a pressure gradient, while the linear contribution arises from the fact that the top plate is moving at a velocity V_1. Figure 3 shows what the velocity profile may look like when just the linear or both linear and parabolic contributions are present. Integrating the velocity profile (5) across the gap gives the volumetric flow rate Q' per unit depth of flow into the page,

$$Q' = \int_0^h v_1 \, \mathrm{d}x_2 = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x_1} + \frac{V_1 h}{2} \qquad (6)$$

3). Steady-state, laminar flow in a cylindrical pipe (Poiseuille flow). This flow is illustrated in Figure 4. Again, we are after an expression for the velocity profile in the pipe. What independent variables does the velocity profile depend on, and why? From the cylindrical version of the differential equation of motion for a Newtonian fluid with constant ρ and μ, with no body forces in the z direction,

$$0 = - \mathrm{d}p/\mathrm{d}z + \mu/r \, \mathrm{d}/\mathrm{d}r \, (r\mathrm{d}v_Z/\mathrm{d}r) \qquad (7)$$

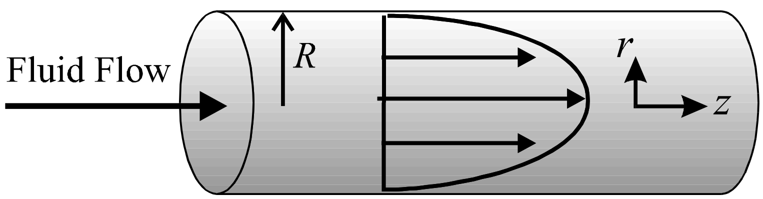

Figure 4.

Assuming that the pressure gradient dp/dz is constant, and applying the boundary conditions $v_1 = 0$ at $r = R$, and dv_Z/dr = 0 at r = 0 (since the flow is symmetric about r, the slope of v_Z with respect to r at r = 0 must be zero) we get

$$v_Z = 1/(4\mu) \, \mathrm{d}p/\mathrm{d}z \, (r^2 - R^2) \qquad (8)$$

Therefore, as for the case of flow between parallel plates, the velocity profile is parabolic. Again, the flow is entirely driven by the presence of a pressure gradient. Note that for a given pressure gradient and distance from the center of the flow, the pipe flow, equation 8, predicts lower velocities than the flow in the gap between parallel plates, equation 4. This is evident because the prefactor $1/4\mu$ in equation 8 is smaller than the prefactor $1/2\mu$ in equation 4. From a physical perspective, why should this be?

Integration of equation 8 over the cross-sectional area of the pipe gives the volumetric flow rate Q,

$$Q = \int_0^R \int_0^{2\pi} v_Z \, r\mathrm{d}\theta \, \mathrm{d}r = -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \qquad (9)$$

External Flows

External flow occurs when a fluid flows over an object. The flow is perturbed due to interaction with the object in its path. Figure 5 illustrates the flow of fluid around a sphere, an example of external flow. Air flow over a house or around a plane is another example of external flow. Often, external flows are turbulent although, for now, we will restrict our attention to laminar external flows. We will discuss external flow around a sphere as our first example.

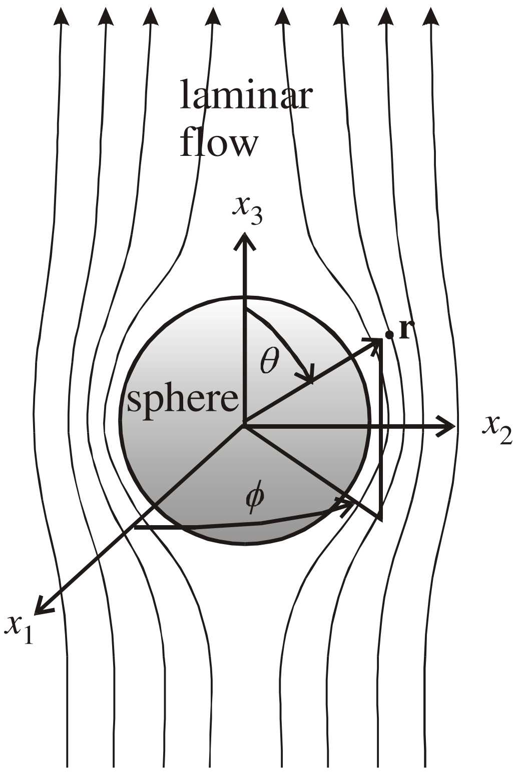

Figure 5.

1). Creeping Flow Around a Sphere. The general problem of flow around a sphere, Fig. 5, is difficult to treat and an analytical solution cannot be found. For the case of so-called "creeping flow", however, an analytical solution is possible. "Creeping" refers to the situation of very small *Re*, when the convective transfer of momentum is insignificant compared to the viscous transfer of momentum. For flow around a sphere *Re* is defined as $Re = \rho V_o D/\mu$ where V_o is the fluid velocity in the x_3 direction far from the sphere, and *D* is the sphere diameter. Small *Re* means that the flow is slow; hence the terminology "creeping."

A famous result that emerges from the solution of the creeping flow around a sphere is that the total force F_D on the sphere due the fluid flowing around it is given by

$$F_D = 6\pi\mu R V_o \qquad (10)$$

where μ is the fluid viscosity, *R* is the radius of the sphere, and V_o is the free stream velocity of the flowing fluid. "Free stream" refers to those parts of the flow that are far from the sphere and are not perturbed by its presence. Equation 10 actually consists of two contributions, the first of which is called the "**form drag**" F_P and is given by

$$F_P = 2\pi\mu R V_o \qquad (11)$$

The form drag arises because the normal stress on the backside (downstream) of the sphere is less than the normal stress on the front side (upstream) of the sphere. The truth of this statement can be directly verified by obtaining the stress distribution from the differential equations of fluid mechanics. The reason why the normal stress is less on the backside of the sphere is in part due to the fact that viscous stresses need to be overcome as the fluid flows from the front to the rear of the sphere; overcoming these viscous stresses is accompanied by a decrease in pressure which results in a decreased normal stress on the backside of the sphere. The important point is that form drag arises from changes in the *normal* stress attributable to the motion of a viscous fluid around the *form* of a body (i.e. the change in normal stress *does not* arise from gravity or some other body force).

The second contribution to the total drag force is termed the "**viscous**", "**skin**", or "**friction**" **drag** F_f,

$$F_f = 4\pi\mu R V_o \tag{12}$$

and is due to the force exerted on the sphere by viscous *shear* stresses acting at the sphere/fluid interface. The sum of the form and friction drags, equations 11 and 12, results in the total drag on the sphere due to the motion of the fluid (equation 10).

Form drag can be reduced by gradually tapering the rear portion of a body (streamlining). An airplane wing is a good example of a streamlined body. However, streamlining increases the surface area of the body, contributing to an increase in its skin friction or viscous drag. Eventually, a decrease in form drag will be more than offset by an increase in viscous drag. The minimization of the total drag force reflects a compromise between these two competing requirements.

2). Boundary Layer Flow over a Flat Plate.

The concept of a boundary layer is extremely important in many applications, both for internal and external flows. Here we introduce it in the context of external flow over a plate, but as discussed at the beginning of this handout boundary layers also play a role in internal flows (i.e. as part of the entry transition region in the pipe flow depicted in Figure 1).

The concept of a boundary layer is useful when discussing flows that may be subdivided into a thin region next to a surface in which viscous effects are important (this region is the "boundary layer"), and an external region in which viscous effects may be neglected (the so-called "potential flow" region). See figure 6. In the potential flow region, viscous effects may be negligible, even if the fluid viscosity is considerable, because the velocity gradients are so small that there is little friction acting between different parts of the fluid. The precise delineation between the boundary layer and the potential flow region is subject to convention; often, the boundary layer thickness is defined by the requirement that the fluid velocity parallel to the surface is equal to *99%* of the free stream velocity V_o. At distances further from the surface than this thickness the flow is viewed as potential flow, while closer to the plate the flow is viewed as boundary layer flow. If the pressure decreases in the direction of the flow (i.e. in Fig. 6, $dp/dx_1 < 0$), so that the pressure gradient favors flow along the surface, the thickness of the boundary layer grows more slowly than if the pressure gradient opposes the flow ($dp/dx_1 > 0$). If the pressure gradient opposes the flow, it may cause the fluid velocity near the surface to reverse direction, and a "separation" of the boundary layer will then occur.

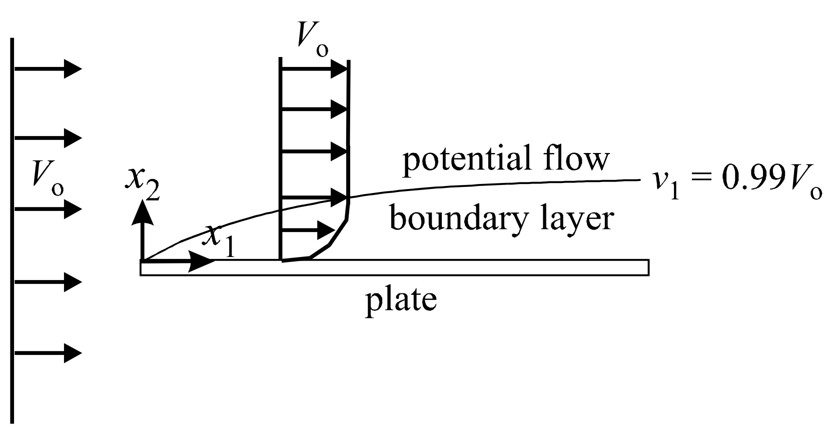

Figure 6.

The "von-Karman" Momentum Integral Equation. We will examine two approaches to analyzing boundary layer flow. The first approach is an example of an **integral approximation method**. The typical purpose of this method is to find a solution for a profile of interest; for example, a velocity profile, a temperature profile, a pressure profile, or a concentration profile. In other words, the usual goal is to determine how velocities, temperature, pressure, or concentrations vary with position or time. Normally, as we know from prior discussion, this objective would be pursued by solving the differential balance laws subject to the appropriate boundary and initial conditions. By solving the differential equations, we would find that profile of interest that satisfies the physical laws underlying the equations (i.e. mass, energy, momentum conservation) at every point. An integral approximation method, however, adopts a different viewpoint. Instead, as you might have guessed from its name, the conservation laws are applied to a control volume; i.e. the solution to the problem will satisfy the conservation laws for a *control volume*, rather than for every point. This methodology does not enforce the conservation laws at all points inside the control volume and the solution found will not, in general, exactly satisfy momentum, energy, or mass conservation at a particular point.

Why use an integral approximation method? The main reason is that it can greatly simplify the mathematics, and still produce reasonably accurate results if applied properly. In the integral approximation method, a **trial function** is assumed for the profile of interest (e.g. the velocity or temperature profile may be assumed to be a 3rd order polynomial). This function will have unknown parameters in it which are determined in the course of solving the problem by requiring that the trial function satisfy the applicable conservation laws for the control volume, and any additional physical constraints (e.g. boundary conditions) that one wishes to enforce. Therefore, the accuracy of the integral approximation approach hinges to a great deal on correct intuition in identifying what physical constraints are most important to satisfy. The mathematics are greatly simplified because, since a trial function is *assumed*, it is not necessary to derive the functional form of the solution from the differential conservation equations. However, a weakness of integral approximation methods is that it is not always obvious how to check the accuracy of the found solution, which may strongly depend on selection of the trial function and decisions as to which physical constraints are enforced.

We will use the integral approximation method to estimate the velocity profile v_1 in the boundary layer over a flat plate, Fig. 6. In turn, the velocity profile will be used to calculate the shear stress exerted on the plate and the thickness of the boundary layer. We start by writing an integral momentum balance for a control volume that encloses a piece of the boundary layer as shown in Fig. 7. The value of v_1 outside of the boundary layer is taken to equal V_o. In other words, we assume that v_1 reaches the exact value V_o of the undisturbed flow at $x_2 = \delta(x_1)$, where $\delta(x_1)$ is the thickness of the boundary layer as a function of the distance x_1 along the plate. In addition to determining a trial function for v_1, we want to develop expressions for the thickness $\delta(x_1)$ and the shear stress $\sigma_{21P}(x_1)$ exerted by the fluid on the plate at $x_2 = 0$ (the subscript "p" indicates that the stress is to be evaluated at the plate).

Considering the control volume in Figure 7, it is evident that we will need the mass flowrates into the control volume across its left, right, and top sides. The flowrates will be needed for writing the integral momentum balance on the control volume. The control volume is chosen to be very thin, of width dx_1, in the x_1 direction. The mass flowrate m^*_L across the left side of the control volume (located at $x_1 = x_L$) is given by

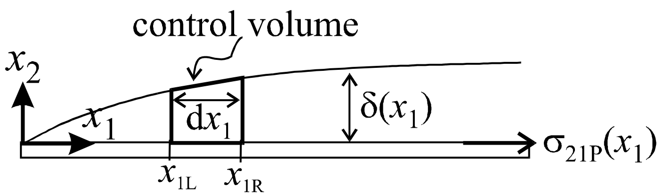

Figure 7.

$$m^*_L = \int_0^W \int_0^\delta \rho v_1 dx_2 dx_3 = W \int_0^\delta \rho v_1 dx_2 \tag{13}$$

where W is the width (into the page, i.e. in the x_3 direction) of the control volume. The mass flowrate m^*_R across the right side of the control volume is

$$m^*_R = -(m^*_L + (\partial m^*_L/\partial x_1)\, dx_1) = -m^*_L - W \frac{\partial}{\partial x_1}\left[\int_0^\delta \rho v_1 dx_2\right] dx_1 \tag{14}$$

Note that the term $(\partial m^*_L/\partial x_1)dx_1$ is simply the change in the mass flowrate m^*_L that occurs over the distance dx_1. Therefore, the flow rate *out* of the control volume at $x_1 = x_R$ is given by $- m^*_L$ (the value at $x_1 = x_L$) plus the term $- (\partial m^*_L/\partial x_1)dx_1$ due to the change in mass flowrate between $x_1 = x_L$ and $x_1 = x_R$. Now, from the integral equation of mass conservation for steady state, there can be no net accumulation of mass in the control volume; therefore, the sum of all the mass flowrates must be zero,

$$m^*_R + m^*_L + m^*_T = 0 \tag{15}$$

where m^*_T is the mass flowrate through the top of the control volume. Inserting equations (13) and (14) into equation (15) and rearranging,

$$m^*_T = +W \frac{\partial}{\partial x_1}\left[\int_0^\delta \rho v_1 dx_2\right] dx_1 \tag{16}$$

Below, we will use the integral mass balance, equation 16, to simplify the integral momentum balance. Since we are interested in shear stress σ_{21P} acting on the plate in the x_1 direction, we need the x_1-component of the momentum balance (note that steady state applies),

$$0 = M_{L1} + M_{R1} + M_{T1} + F_1 \tag{17}$$

where M_{L1}, M_{R1}, and M_{T1} are x_1-momentum convection terms through the left, right, and top sides of the control volume, and F_1 is the total force acting on the control volume in the x_1 direction. Since at steady state there is no accumulation of momentum inside the control volume, the sum of the terms in equation 17 equals zero. The convection terms are:

$$M_{L1} = W \int_0^\delta \rho v_1^2 dx_2 \tag{18a}$$

$$M_{R1} = -(M_{L1} + (\partial M_{L1}/\partial x_1)\, dx_1) = -M_{L1} - W \frac{\partial}{\partial x_1}\left[\int_0^\delta \rho v_1^2 dx_2\right] dx_1 \tag{18b}$$

$$M_{T1} = Vo\, m^*_T = Vo\, W \frac{\partial}{\partial x_1}\left[\int_0^{\delta} \rho v_1 \mathrm{d}x_2\right]\mathrm{d}x_1 \tag{18c}$$

The right hand side of equation (18b) is the value of the x_1-momentum convection *out* of the control volume at $x_1 = x_R$. In equation (18c), the x_1-component of the velocity at the top side of the control volume is equal to V_o, since the top side borders the potential flow region (Figure 7). The product of Vo, which is momentum per mass, with the mass flowrate m^*_T (mass / time) is the rate of momentum convection across the top side of the control volume.

The complete the integral momentum balance, equation 17, we also need to evaluate the forces acting on the control volume along the x_1-direction. These forces include pressure forces plus the shear force between the plate and the fluid (why is there no shear force at the top side of the control volume?). At this stage, an additional assumption is invoked. In particular, it is assumed that the pressure inside the boundary layer is imposed by that existing in the potential flow region just outside the boundary layer. In other words, the pressure inside the boundary layer is assumed to be independent of x_2, so that

$$p_B(x_1) = p_P(x_1) \tag{19}$$

where p_B is the pressure inside the boundary layer and p_P is the pressure in the potential flow region. Equation 19 can be justified by writing out the x_2-component of the differential momentum balance for flow inside the boundary layer. In this differential balance, the velocity component v_2 can be approximated as equal to zero because this velocity component is very small. With $v_2 = 0$, the x_2-component of the differential momentum equation reduces to (you may want to confirm this for yourself)

$$0 = - \partial p/\partial x_2 \tag{19b}$$

In equation 19b, in addition to taking $v_2 = 0$ we have also neglected body forces which is an appropriate approximation if the boundary layer is thin. If $\partial p/\partial x_2 = 0$ then the pressure inside the boundary layer must be independent of x_2, and equation 19b leads directly to equation 19.

In addition, we assume that the pressure in the potential region is constant and so independent of x_1. Equation 19 reduces to

$$p_B = p_P = \text{constant} \qquad \text{(everywhere inside the boundary layer)} \tag{19c}$$

Equation 19c implies that there is no net force, due to pressure, that acts on the control volume in the x_1 direction since the pressure force pushing "right" on the control volume will be exactly counterbalanced by pressure force pushing "left" on the control volume. Then the only contribution to the force F_1 on the control volume is the shear stress against the plate,

$$F_1 = - \sigma_{21p} W \mathrm{d}x_1 \tag{20}$$

where $W\mathrm{d}x_1$ equals the area of the bottom side of the control volume. Note that the shear force exerted by the plate on the control volume is in the negative x_1 direction. Inserting equations 20 and 18a to 18c into equation 17 completes the integral momentum balance, which simplifies to

$$\sigma_{21p} = -\frac{d}{dx_1}\left[\int_0^{\delta} \rho v_1^2 \mathrm{d}x_2\right] + V_o \frac{d}{dx_1}\left[\int_0^{\delta} \rho v_1 \mathrm{d}x_2\right] \tag{21}$$

Combining terms,

$$\sigma_{21p} = \frac{d}{dx_1}\left[\int_0^{\delta} \rho (V_o - v_1) v_1 \mathrm{d}x_2\right] \tag{22}$$

Equation 22 is a statement of momentum conservation, applied to the control volume in Fig. 7. To proceed further (e.g. to evaluate the integral so that the shear stress on the plate can be estimated) we now need to select a trial function for $v_1(x_2)$. The trial function is *assumed* to be represented by a polynomial:

$$v_1/V_o = a + b(x_2/\delta) + c(x_2/\delta)^2 + d(x_2/\delta)^3 \tag{23}$$

In the case of boundary layer flow over a flat plate, the suitability of equation 23 has been verified by experiment and by more detailed calculations (see next section). The four constants *a*, *b*, *c*, and *d* in equation 23 are fixed by imposing four physical constraints:

i). $v_1 = 0$ at $x_2 = 0$ ii). $v_1 = V_o$ at $x_2 = \delta$ iii). $\partial v_1/\partial x_2 = 0$ at $x_2 = \delta$

iv). $\partial^2 v_1/\partial x_2^2 = 0$ at $x_2 = 0$ (24)

Boundary condition (iii) equates the slope of the velocity at the edge of the boundary layer to that in the potential flow region. Boundary condition (iv) is obtained directly from the *differential* momentum balance for the x_1 direction (see equation 14a in handout 8 - dropping out all terms that equal zero at the plate surface leads to boundary condition iv). Applying boundary conditions 24 to equation 23 yields four equations in the four unknowns *a*, *b*, *c*, and *d*. The solution is $a = c = 0$, $b = 3/2$ and $d = -1/2$, so that

$$v_1/V_o = 3/2\ (x_2/\delta) - 1/2\ (x_2/\delta)^3 \tag{25}$$

Higher order polynomials than equation 23 could also have been used to represent the velocity, supplemented by additional boundary conditions (for instance, requiring continuity of higher order derivatives of v_1 at $x_2 = \delta$). However, for the present purposes equation 25 will be sufficient.

Recall that we want to derive expressions for two quantities, σ_{21p} and δ. By definition of the stress tensor for a Newtonian fluid

$$\sigma_{21p} = [\mu\, \mathrm{d}v_1/\mathrm{d}x_2]_{x2=0} = 3/2\ (\mu V_o/\delta) \tag{26}$$

Equation 25 was used in arriving at equation 26. Equation 26 is a constitutive relation which, however, still has two unknowns (σ_{21p} and δ) and so cannot be solved. We therefore supplement equation 26 with the integral momentum balance, equation 22, what leads to two equations in the two unknowns σ_{21p} and δ. Inserting equation 25 into equation 22 and performing the integration results in

$$\sigma_{21\text{p}} = \frac{d}{dx_1}\left(\frac{39}{280}\rho V_o^2 \delta\right)$$

Equating this result to that from equation 26,

$$3/2\ (\mu V_o/\delta) = \frac{39}{280}\rho V_o^2 \frac{d\delta}{dx_1} \quad \rightarrow \quad 420/39\ (\mu V_o)/\rho V_o^2\ \text{d}x_1 = \delta \text{d}\delta \tag{27}$$

Integrating equation 27 (after using the condition $\delta = 0$ at $x_1 = 0$) leads to

$$\delta / x_1 = 4.64/(\rho V_o x_1/\mu)^{1/2} = 4.64/Re_{x1}^{\ 1/2} \tag{28}$$

where Re_{x1} is the Reynold's number defined in terms of the distance x_1 from the plate's leading edge,

$$Re_{x1} = \rho V_o x_1/\mu \tag{29}$$

Equation 28 states that the boundary layer thickness δ is proportional to $x_1^{\ 1/2}$, and is the first of the results we were after. To evaluate $\sigma_{21\text{p}}$, we insert expression 28 into equation 26

$$\sigma_{21\text{p}} = 3/2\ (\mu V_o\ /\ 4.64\ x_1)\ Re_{x1}^{\ 1/2} = 0.323\rho V_o^{\ 2}\ /\ Re_{x1}^{\ 1/2} \tag{30}$$

Equation 30 is the second result we wanted, namely an expression for $\sigma_{21\text{p}}$. Together with equation 28, it represents the key results of the integral approximation method applied to boundary layer flow over a flat plate.

Drag Coefficient. It is customary to define **drag coefficients** C_D for external flows over various objects, including plates. For a particular object, the drag coefficient is defined by

$$C_D = \frac{F_D}{A\dfrac{\rho V_o^2}{2}} \tag{31}$$

where F_D is the total drag force on the object due to viscous stresses and A is an area associated with the object (exact definition of A depends on the type of object). For the case of the plate, A is defined to be the total surface area of *one* side of the plate, so that $A = WL$ where W is the width and L is the length of the plate. If fluid flows both above and below the plate (so that there is a boundary layer both on the top and bottom sides of the plate), the total drag force F_D of the fluid on the plate is:

$$F_D = 2\int_0^W \int_0^L \sigma_{21p}\,\text{d}x_1\text{d}x_3 = 2W \int_0^L 0.323\frac{\rho V_o^2}{\sqrt{Re_{x1}}}\,\text{d}x_1 = 0.646WL\ \rho V_o^{\ 2}\ /\ Re_L^{\ 1/2} \tag{32}$$

In equation 32, $Re_L = \rho V_o L / \mu$. The factor 2 is present because the fluid exerts drag both on top and bottom sides of the plate. Applying the definition of the drag coefficient (equation 31) and using $A = WL$:

$$C_D = 1.29 / Re_L^{1/2} \tag{33}$$

Equation 33 gives the drag coefficient for a flat plate of length L, oriented parallel to the direction of flow, with fluid flowing both above and below the plate. The equation rests on the assumptions of the integral approximation treatment (we will present an alternate derivation in the next section), which include the somewhat arbitrary choice of a polynomial trial function for the velocity profile in the boundary layer. Finally, it is important to keep in mind that we are dealing with flows that are everywhere *laminar*. As will be seen later, the above expressions change when turbulence is present.

The Blasius Solution for Boundary Layer Flow over a Flat Plate. In contrast to the integral approximation approach, the Blasius solution for a laminar boundary layer flow over a flat plate does not make use of integral balances. Rather, it obtains the velocity profile directly from the differential equations of momentum and mass conservation. The assumption of a polynomial form for the velocity profile is thus avoided. We will derive the Blasius solution, and then compare its results to those obtained via the integral approximation approach.

The solution will require us to solve a partial differential equation. In doing so, we will introduce the method of **combination of variables** (also known as the "similarity transform method") to convert the partial differential equation (which will be in two independent variables) into an ordinary differential equation (which will, of course, be in a single independent variable). Combination of variables is often successful when the profile of interest (e.g. of velocity, temperature, pressure, or other field) is to be solved in a semi-infinite medium (i.e. geometrical effects are absent) AND when the profiles at the different positions and/or times of interest are expected to have the "same" shape; i.e. to be *self-similar*. By self-similarity we mean that the profiles could be superimposed on each other by rescaling (i.e. "stretching" or "compressing") the position and/or time variables that the profiles depend on by a characteristic length and/or period.

As in the integral approximation analysis, we will invoke the approximation that the pressure in the boundary layer is imposed by that in the potential flow, using the same justification as described following equation 19. Furthermore, the pressure in the potential flow is again taken to be constant, so that the pressure in the boundary layer is the same everywhere. It can be shown that this approximation holds reasonably well for $Re_{x1} = \rho V_o x_1/\mu > 10000$. The x_1-component of the Navier-Stokes equations for incompressible, constant viscosity flow becomes (again, we neglect body forces)

$$v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = \frac{\mu}{\rho}\left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}\right) \tag{34}$$

An additional simplification arises from stipulating that the dependence of v_1 on x_2 is much stronger than its dependence on x_1. This is a reasonable supposition for a *thin* boundary layer, since v_1 has to change from zero to the free stream value over the small distance corresponding to the thickness of the boundary layer. We can illustrate this better by estimating the magnitudes of the various terms in equation 34. For the estimates, we take the boundary layer length along x_1 to be the length of the plate, thus $\Delta x_1 = L$, and its extent in x_2 as its thickness, $\Delta x_2 = \delta$. The change in v_1 either along the plate (i.e.

from $x_1 = 0$ to $x_1 = L$) or across the boundary layer (i.e. from $x_2 = 0$ to $x_2 = \delta$) is expected to be comparable to V_o. In particular, along the plate $\Delta v_1 = - V_o$, while across the boundary layer $\Delta v_1 = V_o$. Moreover, the magnitude of v_1 at most points in the boundary layer is expected to be similar to V_o. The magnitude of v_2 is expected to be small, close to 0 (since the incoming flow has no x_2 component), but not exactly zero. In fact, v_2 will be positive because, when the fluid hits the plate and is forced to decelerate, there must be a net upward displacement of the flow to satisfy the constraint of incompressibility (for the same reason that we obtain a positive value of m^*_T in equation 16). From the equation of continuity for an incompressible fluid we have

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \qquad \text{therefore} \qquad -\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} \tag{34b}$$

Estimating the terms in equation 34b (also see equation 1 in Handout 9),

$$\frac{V_o}{L} \sim \frac{\Delta v_2}{\delta} \qquad \text{or} \qquad \Delta v_2 = v_2 - 0 = v_2 \sim \delta \frac{V_o}{L} \tag{34c}$$

Therefore, in the boundary layer $v_2 \sim \delta V_o/L$. With the above discussion and the help of equation 34c, we can now estimate the magnitude of each of the terms in the momentum balance, equation 34

$$v_1 \frac{\partial v_1}{\partial x_1} \sim V_o \frac{V_o}{L} \qquad v_2 \frac{\partial v_1}{\partial x_2} \sim \frac{\delta V_o}{L} \frac{V_o}{\delta} = V_o \frac{V_o}{L} \qquad \frac{\partial^2 v_1}{\partial x_1^2} \sim \frac{V_o}{L^2} \qquad \frac{\partial^2 v_1}{\partial x_2^2} \sim \frac{V_o}{\delta^2} \tag{34d}$$

From equations 34d we see that the estimated magnitudes of the two terms on the left of equation 34 are comparable; therefore, we need to keep both of them. On the other hand, the second derivative of v_1 with respect to x_2 is expected to be much greater than its second order derivative with respect to x_1, since $\delta << L$. Therefore, it should be justifiable to drop the $\frac{\partial^2 v_1}{\partial x_1^2}$ term on the right of equation 34, what leads to

$$v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = \frac{\mu}{\rho} \frac{\partial^2 v_1}{\partial x_2^2} \tag{35}$$

Equation 35 is referred to as the "boundary layer equation." In addition to this momentum balance, the solution for the velocity profile in the boundary layer must also satisfy the equation of continuity for an incompressible fluid,

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{36}$$

Finally, we have the boundary conditions, i). $v_1 = 0$ at $x_2 = 0$

ii). $v_2 = 0$ at $x_2 = 0$

iii). $v_1 = V_o$ at $x_2 = \infty$ (37)

The two equations 35 and 36, together with boundary conditions 37, comprise the complete mathematical formulation needed to solve for the two unknowns v_1 and v_2. Before solving for the velocities, it is helpful to introduce the so-called **stream function** ψ, defined as follows:

$$v_1 = -\frac{\partial \psi}{\partial x_2} \tag{38a}$$

$$v_2 = \frac{\partial \psi}{\partial x_1} \tag{38b}$$

The stream function is introduced for mathematical convenience, and can be defined for *any* two-dimensional flow. A two-dimensional flow is one in which two components of the velocity are nonzero, with no flow in the third direction. By defining the velocities in terms of the stream function as in equations 38, the equation of continuity (equation 36) is automatically satisfied since mixed second derivatives of the stream function are equal.

Blasius conjectured that the velocity v_1 depends on x_1 and x_2 through a "combined" variable η only, defined by

$$\eta = C\, x_2 / x_1^{\,n} \tag{39}$$

where C and n are constants. As mentioned earlier, the combination of variables method applies when the profile of interest has a self-similar shape. If v_1 is only a function of η, then the stream function must obey

$$\psi = C'\, x_1^{\,n} F(\eta) \tag{40}$$

where C' is another constant and F is an unknown function of η that needs to be determined. Equation 40 follows from equations 38a, 39, and the requirement that v_1 is only a function of η. We can confirm this last requirement by inserting equation 40 into 38a and applying the chain rule of differentiation,

$$v_1 = -\frac{\partial \psi}{\partial x_2} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x_2} = -\,C'\, x_1^{\,n}\, \mathrm{d}F/\mathrm{d}\eta \;\; C / x_1^{\,n} \; = -\,C'\, C\, \mathrm{d}F/\mathrm{d}\eta \tag{41a}$$

As was stipulated, with the definition 40 for the stream function v_1 depends only on η (note: $\mathrm{d}F/\mathrm{d}\eta$ is a function of η only). We can now also see that the reason for including $x_1^{\,n}$ in equation 40 was to bring about the cancellation of the $x_1^{\,n}$ terms in equation 41a. An expression for v_2 can be derived using equations 40 and 38b,

$$v_2 = n\, C'\, x_1^{\,n-1}\, F \; - n\, C'C \;\; \mathrm{d}F/\mathrm{d}\eta \;\; x_2 / x_1 \tag{41b}$$

Note that v_2 was not postulated to be self-similar; hence it is not a function of η only. In addition,

$$\frac{\partial v_1}{\partial x_2} = -\, C^2 C' / x_1^{\mathrm{n}} \;\; \mathrm{d}^2F/\mathrm{d}\eta^2 \tag{41c}$$

$$\frac{\partial v_1}{\partial x_1} = n\, C' C^2\, x_2 / x_1^{\mathrm{n}+1}\;\; \mathrm{d}^2F/\mathrm{d}\eta^2 \tag{41d}$$

$$\frac{\partial^2 v_1}{\partial x_2^2} = -\, C^3 C' / x_1^{2\mathrm{n}} \;\; \mathrm{d}^3F/\mathrm{d}\eta^3 \tag{41e}$$

Inserting equations 41a to 41e into the momentum balance equation 35, and simplifying,

$$-\, n(C'/C)\, F\, \mathrm{d}^2F/\mathrm{d}\eta^2 = -\, \nu / x_1^{2\mathrm{n}-1}\; \mathrm{d}^3F/\mathrm{d}\eta^3 \tag{42}$$

where $\nu = \mu/\rho$ is the "**kinematic**" **viscosity**.

We have not yet specified the values of n and the constants C and C'. At this stage these parameters are arbitrary, and it makes sense to choose values that will simplify the solution of equation 42. For n we choose

$$n = 1/2 \tag{43}$$

since that will eliminate the x_1 dependence in equation 42, a desirable simplification. The values chosen for C and C' are also arbitrary; a possible choice is

$$C = (V_{\mathrm{o}}/\nu)^{1/2} \text{ and } C' = -\,(V_{\mathrm{o}}\nu)^{1/2} \tag{44}$$

With these choices,

$$\eta = (V_{\mathrm{o}}/\nu)^{1/2}\;\; x_2/x_1^{1/2} \qquad \text{and} \qquad \psi = -\,(V_{\mathrm{o}}\nu)^{1/2}\, x_1^{1/2}\, F(\eta) \tag{45}$$

Also, the momentum balance 42 is rewritten as

$$F\, \mathrm{d}^2F/\mathrm{d}\eta^2 = -\,2\; \mathrm{d}^3F/\mathrm{d}\eta^3 \tag{46}$$

By defining the combined variable η, we have transformed the original partial differential equation 35 into the ordinary differential equation 46, a great simplification. The highest order derivative in equation 46 is third order; therefore, to fully specify the solution to equation 46 three boundary conditions on $F(\eta)$ are needed. These boundary conditions are derived from the original set of boundary conditions listed in equation 37. Using equations 37, 39, 41, 43 and 44, the original boundary conditions are rewritten in terms of F and η as

i). $\mathrm{d}F/\mathrm{d}\eta = 0$ at $\eta = 0$

ii). $F = 0$ at $\eta = 0$

iii). $dF/d\eta = 1$ at $\eta = \infty$ (47)

Equation 46, with the boundary conditions 47, can now be solved for F. Once F is known, v_1 and v_2 can be calculated using equations 41a and 41b. These equations, in terms of $n = 1/2$, $C = (V_o/\nu)^{1/2}$, and $C' = -(V_o\nu)^{1/2}$ become

$$v_1 = V_o\, dF/d\eta \tag{48}$$

$$v_2 = -1/2\,(V_o\nu)^{1/2} x_1^{-1/2} F + 1/2\, V_o\, dF/d\eta\; x_2/x_1 \tag{49}$$

The nonlinear, third order differential equation 46 for F must be solved via numerical techniques. Still, the above effort was not wasted as solving an ordinary differential equation is much easier than solving a partial differential equation.

From the velocity profile $v_1(\eta)$ we can calculate the viscous shear stress σ_{21p} between the fluid and the plate

$$\sigma_{21p} = [\mu\, dv_1/dx_2]_{x2=0} = \mu V_o\, [\, d^2F/d\eta^2\, d\eta/dx_2\,]_{x2=0}$$

$$\sigma_{21p} = \mu V_o\, [\, d^2F/d\eta^2\,]_{x2=0}\, (V_o/\nu)^{1/2} / x_1^{1/2}$$

$$\sigma_{21p} = 0.332\,(\mu/\rho V_o x_1)^{1/2} \rho V_o^2 = 0.332\, \rho V_o^2 / Re_{x1}^{1/2} \tag{50}$$

where $[\, d^2F/d\eta^2\,]_{x2=0} = 0.332$ is obtained from the numerical solution. The chain rule for differentiation was used in the first line leading to equation 50.

Equation 50 can be compared to equation 30 obtained using the integral approximation method

$$\sigma_{21p} = 0.323 \rho V_o^2 / Re_{x1}^{1/2} \tag{30}$$

The two expressions differ just by 2.8 %, due to the slight difference in the numerical prefactors. Therefore, the assumption of a polynomial trial function for the velocity profile in the integral approximation approach was remarkably effective in estimating the viscous drag between the fluid and the plate. As before, we can integrate equation 50 over a plate of length L and width W to get the total frictional drag on the plate,

$$F_D = 2\int_0^W\int_0^L \sigma_{21p}\, dx_1 dx_3 = 2W\int_0^L 0.332 \frac{\rho V_o^2}{\sqrt{Re_{x1}}} dx_1 = 0.664 WL\, \rho V_o^2 / Re_L^{1/2} \tag{51}$$

Recall that F_D, as given in expression 51, applies when fluid flows over both the top and bottom sides of the plate. The drag coefficient, defined by equation 31, is

$$C_D = 1.328 / Re_L^{1/2} \tag{52}$$

Equation 52 can be compared to $C_D = 1.29 / Re_L^{1/2}$ obtained from the integral approximation approach. These expressions for the drag coefficients assume that the flow is *laminar* over the entire length of the plate. We will examine the effects of turbulence in a subsequent handout.

Displacement Thickness. The displacement thickness δ_d, multiplied by the free stream velocity V_o, is defined as the decrease in fluid flow in the x_1 direction (i.e. parallel to the plate) due to the retarding shear stress exerted by the plate on the flowing fluid. Specifically, δ_d is defined by

$$\delta_d V_o = \int_0^\infty V_o \mathrm{d}x_2 - \int_0^\infty v_1 \mathrm{d}x_2 \tag{53}$$

The terms in equation 53 are volumetric flowrates per unit width (into the page) of the plate. The first term on the right is the volumetric flow rate of fluid that would occur in the absence of the plate, and the second term is the actual volumetric flowrate which is less because viscous interaction with the plate retards the flow (i.e. $v_1 < V_o$). The difference between these two flowrates, divided by V_o, gives the displacement thickness,

$$\delta_d = \frac{1}{V_o} \int_0^\infty (V_o - v_1) \mathrm{d}x_2 \tag{54}$$

Since v_1 varies with x_1, the displacement thickness also varies with x_1. In particular, δ_d increases downstream along the plate as the boundary layer thickens. δ_d is occasionally used as an alternate measure of boundary layer thickness.

Momentum Thickness. The momentum thickness δ_M is similar to the displacement thickness, except that it is related to the decrease in the flow of momentum, rather than fluid volume, that arises due to the retarding viscous interactions with the plate. δ_M is calculated from

$$\delta_M = \frac{1}{V_o^2} \int_0^\infty (V_o - v_1)\, v_1 \, \mathrm{d}x_2 \tag{55}$$

δ_M is yet another possible measure of boundary layer thickness.