II. Classical and Relativistic Spacetimes.

*Motivation:* Turing solvable problems require a TM to halt after a finite number of steps with a given output.

- What if we allow TMs to perform an *infinite* number of steps?
- **Then:** Some Turing unsolvable problems may become solvable!
- **But:** How could puny finite humans access the output of a TM that performs an infinite number of steps?

*Answer to come:* Place the puny human in a spacetime that:
- (a) Allows the TM to live to infinity in its rest frame; and
- (b) Allows the human to access the TM's output in a finite amount of time.

- **Mathematically:** A matter of determining the appropriate curved geometry for the given spacetime.
- **Physically:** Are such appropriately curved spacetimes physically possible?
1. Types of Spacetimes

- A *spacetime* is a 4-dim collection of points with *additional structure*.

> Typically, one or more metrics = a specification of the spatial and temporal distances between points.
Two ways spacetimes can differ:

(1) Different ways of specifying distances between points yield different types of spacetimes.

- *Classical spacetimes* have *separate* spatial and temporal metrics: only one way to split time from space (spatial and temporal distances are *absolute*).

- *Relativistic spacetimes* have a *single* spatiotemporal metric, and how it gets split into spatial and temporal parts depends on one's inertial reference frame (spatial and temporal distances are *relative*).
(2) Metrics can be flat or curved: how one specifies the distance between points encodes the curvature of the spacetime.

- Classical spacetimes can be flat or curved.
- Relativistic spacetimes can be flat (Minkowski spacetime) or curved (general relativistic spacetimes).

- Two ways curvature can manifest itself:

The spatial slices can be flat or curved.

How the spatial slices are "rigged" together can be flat or curved.
2. Classical Spacetimes

*Newtonian spacetime* is a 4-dim collection of points such that:

(N1) Between any two points $P(t, x, y, z)$, $Q(t + Δt, x + Δx, y + Δy, z + Δz)$ there is a definite *temporal interval* $Δt$.

(N2) Between any two points $P(t, x, y, z)$, $Q(t + Δt, x + Δx, y + Δy, z + Δz)$ there is a definite *Euclidean distance* $Δs$,

$$Δs = \sqrt{(Δx)^2 + (Δy)^2 + (Δz)^2}$$

- Consequences of (N1) and (N2):
  
  (a) All worldlines have a definite *absolute velocity* $\frac{Δs}{Δt}$.

(b) There is a privileged collection of worldlines defined by $\frac{Δs}{Δt} = 0$.

  - This defines a *single, privileged, frame of reference* (absolute space).

(c) All worldlines have a definite *absolute acceleration* $\frac{d}{dt}\left(\frac{Δs}{Δt}\right)$.

*But absolute space and absolute velocity are unobservable!*
Neo-Newtonian spacetime is a 4-dim collection of points such that:

(NN1) Between any two points \( P(t, x, y, z) \), \( Q(t + \Delta t, x + \Delta, y + \Delta y, z + \Delta z) \) there is a definite temporal interval \( \Delta t \),

(NN2) Between any two simultaneous points \( P(t, x, y, z) \), \( Q(t, x + \Delta x, y + \Delta y, z + \Delta z) \) there is a definite Euclidean distance \( \Delta s \),

\[
\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.
\]

(NN3) Any worldline \( O \) through a given point \( P \) has a definite curvature.

- Consequences of (NN2):
  - No definite distance between points at different times on any worldline \( O \).
  - So no definite velocity for any worldline: velocity is relative!
  - So no single privileged frame of reference (no absolute space).

- Consequence of (NN3): acceleration remains absolute!
1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

1. Single, privileged inertial frame.
2. Velocity is absolute.
3. Acceleration is absolute.
4. Simultaneity is absolute.

- Both Newtonian spacetime and Neo-Newtonian spacetime have absolute temporal metrics: Everyone agrees on what time it is.
- Relativistic spacetimes have no absolute temporal metric: What time it is depends on your inertial reference frame.
3. Relativistic Spacetimes

- Light Postulate of Special Relativity entails:
  - The speed of light $c$ is the same in all inertial reference frames.

- $O'$ is moving at constant velocity with respect to $O$.
- $O$ and $O'$ must measure same speed $c$ for light signal.
- **So:** The $x'$ axis must be inclined by the same amount $\theta$ from the $x$ axis that the $t'$ axis is inclined from the $t$ axis.
- **Thus:** $O$ and $O'$ must disagree on spatial and temporal measurements!
3. Relativistic Spacetimes

- Light Postulate of Special Relativity entails:
  - The speed of light $c$ is the same in all inertial reference frames.

- $O$ and $O'$ make different judgements of simultaneity (relativity of simultaneity).
- $P$ and $Q$ are simultaneous according to $O'$.
- $P$ happens before $Q$ according to $O$.

*The speed of light $c$ is the same in all inertial reference frames.*
**Spacetime of Special Relativity = Minkowski spacetime**

*Minkowski spacetime* = 4-dim collection of points such that between any two points \( P( t, x, y, z) \), \( Q(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z) \) there is a definite spacetime interval given by

\[
\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.
\]

- Similar to Euclidean *spatial* interval \( \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \).
- **But**: Includes the time coordinate difference, too! And it's *negative*!
- **Idea**: All inertial frames will agree on the spatiotemporal distance \( \Delta s \) between any points \( P \) and \( Q \).
- But they will disagree on how \( \Delta s \) gets split into a temporal part and a spatial part: they will disagree on measurements of time and measurements of space.
• All inertial frames agree on the *spacetime* distance between any two points $P$ and $Q$.

• They will disagree on the *temporal* distance between $P$ and $Q$ (time dilation) and on the *spatial* distance (length contraction).

• They will *disagree* on how they split the spacetime distance into temporal and spatial parts.

\[
\Delta s = \sqrt{-(c\Delta t)^2 + (\Delta x)^2} = \sqrt{-(c\Delta t')^2 + (\Delta x')^2}
\]
• The Minkowski spacetime interval is encoded in the Minkowski metric $\eta_{\mu\nu}$.

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu, \quad \mu, \nu = 0, 1, 2, 3$$

• Infinitesimally: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

\[ \Delta x^0 = c\Delta t, \Delta x^1 = \Delta x, \quad \Delta x^2 = \Delta y, \Delta x^3 = \Delta z, \]
\[ \eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

**Three general forms of $(\Delta s)^2$:**

- **Timelike interval:** $\Delta t > \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \Rightarrow (\Delta s)^2 < 0$
- **Lightlike interval:** $\Delta t = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \Rightarrow (\Delta s)^2 = 0$
- **Spacelike interval:** $\Delta t < \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \Rightarrow (\Delta s)^2 > 0$

• All inertial frames agree on whether a given interval $(\Delta s)^2$ is less than, equal to, or greater than 0.

• **Thus:** There are three types of worldline in Minkowski spacetime: timelike worldlines, lightlike worldlines, and spacelike worldlines.
• **Hence:** The Minkowski metric defines a *lightcone* at any point $P$:

- **Lightlike worldline** $\Rightarrow \Delta t = \Delta x$ (objects with speeds $= c$)
- **Timelike worldline** $\Rightarrow \Delta t < \Delta x$ (objects with speeds $< c$)
- **Spacelike worldline** $\Rightarrow \Delta t > \Delta x$ (objects with speeds $> c$)
**Neo-Newtonian Spacetime**

1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is absolute.

**Minkowski Spacetime**

1. Many inertial frames; none privileged.
2. Velocity is relative.
3. Acceleration is absolute.
4. Simultaneity is relative.
5. Invariant light-cone structure at each point.
**Problem:** Special relativity does not account for the gravitational force.

- To include gravity...

  Geometricize it! Make it a feature of spacetime geometry.

**Two requirements:**

1. New theory ("general relativity") must reduce to special relativity in sufficiently flat regions of spacetime:

   - Replace $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ with $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

   - Flat Minkowski metric $\eta_{mn}$

   - Non-flat metric $g_{\mu\nu}$

2. Require $g_{\mu\nu}$ to reduce to $\eta_{mn}$ in small regions of spacetime.

Any sufficiently small piece looks flat

arbitrarily curved surface
**Problem**: Special relativity does not account for the gravitational force.

- To include gravity...

  ![Geometricize it! Make it a feature of spacetime geometry.](image)

**Two requirements**:

(2) Curvature of spacetime must be related to matter density:

- The Einstein equations (1916):

  \[ G_{\mu\nu} = \kappa T_{\mu\nu} \]

  *Einstein tensor encodes curvature of spacetime as a function of \( g_{\mu\nu} \)*  
  *Stress-energy tensor encodes matter density*

- **Consequence**: The Minkowski metric is the solution for zero curvature \( G_{\mu\nu} = 0 \) (*i.e.*, spatiotemporal flatness).
A general relativistic spacetime = 4-dim collection of points such that between any two points (infinitesimally close) points, there is a definite spacetime interval given by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ is a Lorentzian metric that satisfies the Einstein equations.

"reduces to the Minkowski metric at any point"
Invariant light-cone structure at each point (i.e., light-cones all have same size and orientation).

Light-cone structure at each point is not invariant: light-cones can twist and turn due to curvature.

- **Idea**: The light-cone structure constrains the motion of physical objects (traveling on timelike worldlines).

- **And**: In an arbitrary general relativistic spacetime, the matter density determines the light-cone structure.