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# Chapter 3

# Spacetime Structuralism

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## Abstract

In this essay, I consider the ontological status of spacetime from the points of view of the standard tensor formalism and three alternatives: twistor theory, Einstein algebras, and geometric algebra. I briefly review how classical field theories can be formulated in each of these formalisms, and indicate how this suggests a structural realist interpretation of spacetime.

## 1. Introduction

This essay is concerned with the following question: If it is possible to do classical field theory without a 4-dimensional differentiable manifold, what does this suggest about the ontological status of spacetime from the point of view of a semantic realist? In Section 2, I indicate *why* a semantic realist would want to do classical field theory without a manifold. In Sections 3–5, I indicate the extent to which such a feat is possible. Finally, in Section 6, I indicate the type of spacetime realism this feat suggests.

## 2. Manifolds and manifold substantivalism

In classical field theories presented in the standard tensor formalism, spacetime is represented by a differentiable manifold $M$ and physical fields are represented by tensor fields that quantify over the points of $M$. To some authors, this has suggested an ontological commitment to spacetime points (e.g., Field, 1989; Earman, 1989). This inclination might be seen as being motivated by a general semantic realist desire to take successful theories at their face value, a desire for a *literal interpretation* of the claims such theories make (Earman, 1993; Horwich, 1982). Arguably, the most literal interpretation of classical field theories motivated in this way is manifold substantivalism. Manifold substantivalism consists of two claims.

(i) *Substantivalism*: Manifold points represent real spacetime points.
(ii) *Denial of Leibniz Equivalence*: Diffeomorphically related models of classical field theories in the tensor formalism represent distinct physically possible worlds.

Both claims can be motivated by a literal interpretation of the manifold $M$ that appears in such theories. Claim (i) follows, as suggested above, from a literal construal of tensor fields defined on $M$, and claim (ii) follows from a literal construal of $M$ as a set of *distinct* mathematical points. Unfortunately for the semantic realist, however, manifold substantivalism succumbs to the hole argument, and while spacetime realists have been prolific in constructing versions of spacetime realism that maneuver around the hole argument, all such versions subvert in one form or another the semantic realist's basic desire for a literal interpretation[1]. But what about interpretations of classical field theories formulated in formalisms in which the manifold does not appear? Perhaps spacetime realism can be better motivated in such formalisms while at the same time remaining true to its semantic component.

As a concrete example, consider classical electrodynamics (*CED*) in Minkowski spacetime. Tensor models of *CED* in Minkowski spacetime are given by $(M, \eta_{ab}, \partial_a, F_{ab}, J_a)$, where $M$ is a differentiable manifold, $\eta_{ab}$ is the Minkowski metric, $\partial_a$ is the derivative operator associated with $\eta_{ab}$, and $F_{ab}$ and $J_a$ are tensor fields that represent the Maxwell field and the current density and that satisfy the Maxwell equations.

$$\eta_{ab}\partial_a F_{bc} = 4\pi J_c, \qquad \partial_{[a}F_{bc]} = 0 \tag{1}$$

This suggests that $M$ plays two roles in tensor formulations of classical field theories.

(a) A *kinematical* role as the support structure on which tensor fields are defined. In this role, $M$ provides the mathematical wherewithal for representations of physical fields to be defined.

[1]For a quick review of the hole argument and positions staked out in the literature, see Bain (2003). Spacetime realists who adopt (i) but deny (ii) ("sophisticated substantivalists") give up the semantic realist's desire for a literal interpretation of manifold points and subsequently have to engage in metaphysical excursions into the notions of identity and/or possibility.

(b) A *dynamical* role as the support structure on which derivative operators are defined. In this role, $M$ provides the mathematical wherewithal for a dynamical description of the evolution of physical fields in the form of field equations.

To do away with $M$ and still be able to do classical field theory, an alternative formalism must address both of these roles. In particular, it must provide the means of representing classical fields, and it must provide the means of representing the dynamics of classical fields.

## 3. Manifolds vs. twistors

In this section, I indicate that for certain conformally invariant classical field theories, the twistor formalism is expressively equivalent to the tensor formalism. Standard examples of such theories include

(a) fields that describe geodesic, shear-free, null congruences;
(b) zero rest mass fields;
(c) anti-self-dual Yang-Mills fields; and
(d) vacuum solutions to the Einstein equations with anti-self-dual Weyl curvature.

I indicate how these results follow from a general procedure known as the Penrose Transformation, and discuss their extensions and limitations[2]. I suggest that the concept of spacetime that arises for such field theories is very different, under a literal interpretation, from the one that arises in the tensor formalism.

The twistor formalism rests on a correspondence between complex, compactified Minkowski spacetime $\mathbb{CM}^c$ and a complex projective 3-space referred to as projective twistor space $\mathbb{PT}$. One way to initially understand this correspondence is to first note that compactified Minkowski spacetime $\mathbb{M}^c$ is the carrying space for matrix representations of the 4-dimensional conformal group

[2]The limitation to conformally-invariant field theories will be discussed below at the end of Section 3.1. For some initial motivation, the conceptual significance of example (a), for instance, is that spacetimes that admit geodesic, shear-free, null congruences are *algebraically special* (technically, one or more of the four principle null directions of the Weyl curvature tensor of such spacetimes coincide). Whether or not there is physical significance associated with this mathematical constraint, it does allow solutions to the Einstein equations to be more readily constructed. For instance, the Kerr solution that describes a charged, rotating black hole is algebraically special.

$C(1, 3)$, comprised of conformal transformations on Minkowski spacetime[3]. Next note that (non-projective) twistor space $\mathbb{T}$ is the carrying space for matrix representations of $SU(2, 2)$, which is the double-covering group of $SO(2, 4)$, which itself is the double-covering group of $C(1, 3)$. Hence twistor space encodes the *conformal structure* of Minkowski spacetime, and the twistor correspondence will allow us to rewrite *conformally invariant* field theories in terms of twistors. The precise correspondence requires the complexification of $\mathbb{M}^c$ and the extension of $\mathbb{T}$ to projective twistor space $\mathbb{PT}$. To get a feel for the latter, note that $\mathbb{T}$ can be defined as the space of solutions $(\omega^A, \pi_{A'}) \equiv Z^\alpha (\alpha = 0, 1, 2, 3)$ of the *twistor* equation $\nabla_{B'}{}^B \omega_C(x) = -i\varepsilon_B{}^C \pi_{A'}$, a general solution having the form $\omega^A(x) = \omega_0^A - ix^{AA'}\pi_{A'}$, where $\omega_0^A$ and $\pi_{A'}$ are constant 2-spinors[4]. So-defined, $\mathbb{T}$ is a 4-dimensional complex vector space with a Hermitian 2-form $\sum_{\alpha\beta}$ (a "metric") of signature $(+ + - -)$, and one can then show that it carries a matrix representation of $SU(2, 2)$. $\mathbb{PT}$ is then the 3-complex-dimensional space of 2-spinor pairs $(\omega^A, \pi_{A'})$, up to a complex constant, that satisfy the twistor equation. Under this initial understanding, a twistor $Z^\alpha$ is nothing but a particular "spacetime-indexed" pair of 2-spinors. However, as will be noted below, there are a number of other ways to interpret twistors.

To reiterate, the twistor correspondence allows solutions to certain conformally invariant hyperbolic differential equations in Minkowski spacetime to be encoded in complex-analytic, purely geometrical structures in an appropriate twistor space. Hence, the dynamical information represented by the differential equations in the tensor formalism gets encoded in geometric structures in the twistor formalism. Advocates of the twistor formalism emphasize this result — they observe that, in the twistor formalism, there are no dynamical equations; there is just geometry. This suggests that a naive semantic realist may be faced with a non-trivial task in providing a literal interpretation of classical field theories in the twistor formalism. Before discussing this task, I will briefly

[3]Conformal transformations preserve angles but not necessarily lengths. In Minkowski spacetime $(M, \eta_{ab})$ they preserve the Minkowski metric $\eta_{ab}$ up to scale (i.e., they map $\eta_{ab} \mapsto \Omega^2 \eta_{ab}$, where $\Omega = \Omega(x)$ is a smooth, positive scalar function on $M$) and consist of Poincaré transformations $x^a \mapsto \Lambda_b^a x^b + r^a$, dilations $x^a \mapsto kx^a$, and inversions $x^a \mapsto (y^a - x^a)/(y_b - x_b)(y^b - x^b)$, where $\Lambda_b{}^a$, $r^a$, and $k$ are constant. Inversions are singular at points on the light cone centered at $y^a$. To construct a carrying space that includes inversions, $M$ is compactified by attaching a boundary $\mathscr{I} = \partial M$ consisting of a light cone at infinity. Inversions then interchange $\mathscr{I}$ with the light cone at $y^a$. We thus have $\mathbb{M}^c = (\mathscr{I} \cup M, \eta_{ab})$.

[4]Recall that the 2-spinor $\omega^A$ is an element of a complex 2-dimensional vector space $\mathbb{S}$ endowed with a bilinear anti-symmetric 2-form (the spinor "metric") $\varepsilon_{AB}$. $\mathbb{S}$ is the carrying space for representations of the group, $SL(2, \mathbb{C})$, which is the double-covering group of the Lorentz group $SO(1, 3)$. The 2-spinor $\pi_{A'}$ is an element of the Hermitian conjugate vector space $\mathbb{S}'$. (Here and below the abstract index notation for 2-spinors and for tensors is used. In particular, 2-spinor indices are raised and lowered via the metrics $\varepsilon_{AB}$, $\varepsilon_{A'B'}$, and tensor indices $b$ can be exchanged for pairs of spinor indices $BB'$.)

describe the mathematics underlying the twistor correspondence and its application to classical field theories[5].

### 3.1. The twistor correspondence and the Penrose transformation

The twistor correspondence can be encoded most succinctly in a double fibration of a correspondence space $\mathbb{F}$ into $\mathbb{CM}^c$ and $\mathbb{PT}$ (see, e.g., Ward & Wells, 1990, p. 20). In fiber bundle lingo, such a construction consists of two base spaces that share a common bundle space[6]. This common bundle space then allows structures in one base space to be mapped onto structures in the other. In the case in question, the common bundle space $\mathbb{F}$ is given by the primed spinor bundle over $\mathbb{CM}^c$ consisting of pairs $(x^a, \pi_{A'})$ where $x^a$ is a point in $\mathbb{CM}^c$ and $\pi_{A'}$ is a primed 2-spinor. The double fibration takes the explicit form,

$$\begin{array}{ccccc} & & \mathbb{F} & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ \mathbb{PT} & & & & \mathbb{CM}^c \end{array} \qquad \begin{array}{ccccc} & & \textit{correspondence space} & & \\ & \swarrow & & \searrow & \\ \textit{twistor space} & & & & \textit{spacetime} \end{array}$$

where the projection maps $\mu$, $v$ are given by

$$v : (x^a, \pi_{A'}) \to x^a$$

$$\mu : (x^a, \pi_{A'}) \to (\mathrm{i}x^{AA'}\pi_{A'}, \pi_{A'})$$

These maps are constructed so that they give the correspondence between elements of $\mathbb{CM}^c$ (complex spacetime points) and elements of $\mathbb{PT}$ (projective twistors) by the following relation

$$\omega^A = \mathrm{i}x^{AA'}\pi_{A'} \qquad (KC)$$

known as the Klein correspondence[7]. It expresses the condition for the twistor $(\omega^A, \pi_{A'}) \in \mathbb{T}$ to be incident with the point $x^a \in \mathbb{CM}^c$[8]. Based on this correspondence,

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[5]What follows is an exposition of what has been informally called ‘‘Stone-Age’’ twistor theory (twistor theory during the period 1967–1980). ‘‘21st Century’’ twistor theory has advanced quite a way from $\mathbb{CM}^c$ with current applications in such far-flung areas as string theory (Witten, 2004) and condensed matter physics (Sparling, 2002).

[6]In fiber bundle theory, a bundle space consists of algebraic objects (the ‘‘fibers’’) that are parameterized by the points of a base space. Intuitively, the bundle space lives over the base space and consists of fibers, one for each point of the base space, that are woven together in a smooth way.

[7]So-named for a construction in algebraic geometry that was first given by F. Klein in 1870 (‘‘Zur Theorie der Liniercomplexe des ersten und zweiten Grades’, *Math. Ann.* **2**, 198). Klein demonstrated that the points of a 4-dimensional quadric surface embedded in a 6-dimensional space can be put in 1–1 correspondence with the lines of a projective 3-space. Penrose (1967) introduced the twistor formalism based on the related observation that compactified Minkowski spacetime $\mathbb{M}^c$ can be viewed as a 4-quadric surface embedded in the projective 5-space associated with the 6-dimensional carrying space of representations of $SO(2, 4)$.

[8]$(KC)$ gives the locus of points in $\mathbb{CM}^c$ where solutions to the twistor equation vanish.

Table 1
Geometrical correspondences between projective twistor space and complex compactified Minkowski spacetime

| $\mathbb{PT}$ | $\mathbb{CM}^c$ |
|---|---|
| Point | α-plane |
| Line | Point |
| Point in $\mathbb{PN}$ | Real null geodesic |
| Point in $\mathbb{PT}^+ \cup \mathbb{PT}^-$ | Real Robinson congruence |
| Line in $\mathbb{PN}$ | Real point |
| Intersection of lines | Null separation of points |

the maps allow structures in $\mathbb{PT}$ to be pulled up to $\mathbb{F}$ and then pushed down to $\mathbb{CM}^c$, and vice versa. In particular, the copy in $\mathbb{PT}$ of the fiber $v^{-1}(x^a)$ is obtained directly from (*KC*) by holding $x^{AA'}$ fixed and varying $(\omega^A, \pi_{A'})$. One obtains a complex linear 2-dimensional space in $\mathbb{T}$, which defines a line in $\mathbb{PT}$. The copy in $\mathbb{CM}^c$ of the fiber $\mu^{-1}(\omega^A, \pi_{A'})$ is obtained in a similar manner by holding the twistor $(\omega^A, \pi_{A'})$ fixed and varying the spacetime point $x^{AA'}$. This defines a complex null 2-dimensional plane in $\mathbb{CM}^c$ referred to as an α-plane. Hence under (*KC*), points in $\mathbb{CM}^c$ (complex spacetime points) correspond to "twistor lines", and points in $\mathbb{PT}$ (projective twistors) correspond to α-planes. A summary of similar geometrical correspondences is given in Table 1[9].

We have thus obtained the points of $\mathbb{CM}^c$ from twistors. But to do field theory, we need more than just manifold points: we need fields and derivative operators. More precisely, we need to identify those field-theoretic structures in $\mathbb{CM}^c$ that can be pulled up to $\mathbb{F}$ and then pushed down to $\mathbb{PT}$. A number of results in the twistor literature indicate the extent to which such an identification is possible. These results collectively are referred to as the *Penrose Transformation*. Each establishes a correspondence between purely geometrical/topological structures in an appropriate twistor space and the solutions to particular field equations in spacetime. These results can be divided overall into two categories.

(A) Those that are based on the double fibration between $\mathbb{PT}$ and $\mathbb{CM}^c$. ("Flat" twistor theory.)
(B) Those that are based on a structurally similar double fibration in which $\mathbb{CM}^c$ is replaced by a curved manifold. ("Curved" twistor theory.)

[9]For details see, e.g., Huggett and Todd (1994, pp. 55–58). In Table 1, $\mathbb{PT}^+$, $\mathbb{PT}^-$, and $\mathbb{PN}$ are regions of $\mathbb{PT}$ defined by $Z^\alpha \bar{Z}_\alpha > 0$, $Z^\alpha \bar{Z}_\alpha < 0$, and $Z^\alpha \bar{Z}_\alpha = 0$, respectively, where $\bar{Z}_\alpha$ is the dual twistor defined by the Hermitian 2-form on $\mathbb{T}$ : $\bar{Z}_\alpha = \sum_{\alpha\beta} Z^\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$, where the bar is complex conjugation. A *Robinson congruence* is a family of null geodesics that twist about each other (the origin of the term "twistor").

There are three important results under (A): Kerr's Theorem, The Zero Rest Mass Penrose Transformation (*ZRMPT*), and Ward's Theorem; and one primary result under (B): The Non-linear Graviton Penrose Transformation (*NGPT*). In the remainder of this section, I will state each without proof and briefly describe its content.

**(A1) Kerr's Theorem.** Let $Q$ be a holomorphic surface in $\mathbb{PT}$; i.e., defined by $f(Z^\alpha) = 0$, for some homogeneous holomorphic function $f(Z^\alpha)$. Then the intersection of $Q$ with $\mathbb{PN}$ defines an analytic shear-free congruence of null geodesics in $\mathbb{M}^c$. Conversely, an analytic shear-free null congruence in $\mathbb{M}^c$ defines the intersection of $\mathbb{PN}$ with a holomorphic surface $Q$ given by the zero locus of an arbitrary homogeneous holomorphic function $f(Z^\alpha)$[10].

*Comments.* For a proof, see Huggett and Todd (1994, p. 60). An analytic shear-free null congruence in $\mathbb{M}^c$ is given by a spinor field $o^A$ satisfying $o^A o^B \partial_{BB'} o_A = 0$. Kerr's Theorem thus states that such spinor fields in $\mathbb{M}^c$ correspond to the intersections of surfaces in $\mathbb{PT}$.

**(A2) Zero Rest Mass Penrose Transformation (ZRMPT).**

$$H^1(\mathbb{PT}^+; \mathcal{O}(-n-2)) \cong \{ZRM \text{ fields } \varphi_{A'\ldots B'}(x) \text{ of helicity } n \text{ holomorphic on } \mathbb{CM}^+\}.$$

$$H^1(\mathbb{PT}^-; \mathcal{O}(n-2)) \cong \{ZRM \text{ fields } \phi_{A\ldots B}(x) \text{ of helicity } -n \text{ holomorphic on } \mathbb{CM}^-\}.$$

*Comments.* For a proof, see Huggett and Todd (1994, pp. 91–98). *ZRMPT* states two isomorphisms. First the objects on the left: Here, for instance, $H^1(\mathbb{PT}^+; \mathcal{O}(-n-2))$ is the first cohomology group of $\mathbb{PT}^+$ with coefficients in $\mathcal{O}(-n-2)$, the sheaf of germs of holomorphic functions of homogeneity $-n-2$ over $\mathbb{PT}^+$[11]. The elements of $H^1(\mathbb{PT}^+; \mathcal{O}(-n-2))$ consist of equivalence classes $[f]$ of homogeneous functions of degree $-n-2$ defined on the intersections $U_i \cap U_j$ of a given open cover $\{U_i\}$ of $\mathbb{PT}^+$. Two elements $f_{ij}$, $g_{ij}$, of $[f]$ are equivalent *iff* they differ by a coboundary: $f_{ij} - g_{ij} = h_{ij}$, where $\delta h_{ij} = 0$ for the coboundary map $\delta$. Next, the objects on the right: zero rest mass (*ZRM*) fields are fields (here represented by spinor fields) that satisfy the zero rest mass field equations: $\partial^{AA'} \phi_{A'_{\ldots B'}}(x) = 0$, and $\partial^{AA'} \varphi_{A_{\ldots B}}(x) = 0$, where the number of indices corresponds to twice the spin/helicity. Hence, *ZRMPT* again establishes a correspondence between geometric (topological) objects in $\mathbb{PT}$ and fields satisfying a dynamical field equation in $\mathbb{CM}^c$.

[10] $f(Z^\alpha)$ is holomorphic if it satisfies the Cauchy–Riemann equations: $\partial f / \partial \bar{Z}^\alpha = 0$. $f(Z^\alpha)$ is homogeneous of degree $k$ if $Z^\alpha(\partial f / \partial \bar{Z}^\alpha) = kf$.

[11] A *sheaf* over a topological space $X$ assigns a type of algebraic object to every open set $U$ of $X$. (Compare with a fiber bundle over $X$, which assigns an object to every point of $X$.) The cohomology "group" $H^1(\mathbb{PT}^+; \mathcal{O}(-n-2))$ is really a module over the ring defined by $\mathcal{O}(-n-2)$, i.e., it is a "slightly relaxed" vector space with vectors in $\mathbb{PT}^+$ and scalars in $\mathcal{O}(-n-2)$.

**(A3) Ward's Theorem.** Let $U$ be an open region in $\mathbb{CM}^c$ and $U'$ the corresponding region in $\mathbb{PT}$ under (*KC*), which maps points $x \in \mathbb{CM}^c$ into lines $L_x \subset \mathbb{PT}$. There is a 1–1 correspondence between

(a) anti-self-dual $GL(n, \mathbb{C})$ Yang-Mills gauge fields $F_{ab}$ on $U$; and
(b) rank $n$ holomorphic vector bundles $B$ over $U'$, such that the restriction $B|_{Lx}$ of $B$ to the line $L_x \subset U'$ is trivial for all $x \in U$.

*Comments*. For a proof, see Ward and Wells (1990, pp. 374–381). Ward's Theorem states that an anti-self-dual[12] Yang-Mills gauge field on $\mathbb{CM}^c$ is equivalent to a holomorphic vector bundle over $\mathbb{PT}$ which is trivial (i.e., constant) on twistor lines. For $n = 1$, one obtains an anti-self-dual Maxwell field as a complex line bundle on $\mathbb{PT}^+$. This is a non-linear version of the *ZRMPT* $n = 1$ case.

The twistor correspondences (A1–A3) are for flat spacetimes (in particular, for $\mathbb{CM}^c$). The extension to curved spacetimes is non-trivial. It turns out that solutions to the twistor equation are constrained by the condition $\Psi_{ABCD}\omega^D = 0$, where $\Psi_{ABCD}$ is the Weyl conformal curvature spinor. Hence, twistors are primarily only well defined in *conformally flat* ($\Psi_{ABCD} = 0 = \bar{\Psi}_{A'B'C'D'}$) spacetimes[13]. One way to circumnavigate this "obstruction" is to complexify the spacetime and impose the conditions $\bar{\Psi}_{A'B'C'D'} = 0$ and $\Psi_{ABCD} \neq 0$[14]. This entails that the Weyl tensor is anti-self-dual, hence such a spacetime $\mathscr{M}$ is referred to as anti-self-dual (or right-conformally flat). Such an $\mathscr{M}$ has a globally well-defined

[12]A Yang-Mills field $F_{ab}$ is anti-self-dual just when it satisfies $*F_{ab} = -iF_{ab}$, where * is the Hodge-dual operator. The theorem rests primarily on the fact that $F_{ab}$ is anti-self-dual *if and only if*, for every α-plane $Z'$ that intersects $U$, the restriction of the covariant derivative $\nabla_a$ to $U \cap Z'$ satisfies $n_a \nabla_a \psi = 0$, for any vector field $n_a$ tangent to $Z'$ and any section $\psi$ of the vector bundle associated with $F_{ab}$. Put simply, $F_{ab}$ is anti-self-dual *if and only if* its associated covariant derivative $\nabla_a = \partial_a - ieA_a$ is flat on α-planes.

[13]The Weyl conformal curvature tensor $C_{abcd}$ is the trace-free, conformally invariant part of the Riemann curvature tensor. Its 2-spinor equivalent is $C_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}$. Solutions to the twistor equation also exist in (algebraically special type IV) spacetimes in which the Weyl spinor is null; i.e., can be given by $\Psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D$, for some non-vanishing $\alpha_A$.

[14]This cannot be done in real spacetimes in which the primed and unprimed Weyl spinors are complex conjugates of each other. The move to complex spacetimes removes the operation of complex conjugation allowing both quantities to be treated independently. For details, see Penrose and Ward (1980). They also review two alternative ways to address the obstruction by considering twistors at a point on each null geodesic ("local twistors"), or twistors defined relative to hypersurfaces ("hypersurface twistors"). For the latter, when the null cone at infinity is chosen as the hypersurface, the resultant structures are known as asymptotic twistors. These approaches seem problematic in the context of the present essay insofar as they define twistors relative to structures defined on a pre-existing spacetime manifold. Recently, Sparling (1998) has introduced negative rank differential forms as another means of addressing the obstruction.

family of α-planes, hence a corresponding (projective) twistor space $\mathbb{P}\mathcal{T}$ can be constructed. Schematically, we then have the following double fibration.

$$\begin{array}{ccc} & \mathbb{F} & \\ \swarrow & & \searrow \\ \mathbb{P}\mathcal{T} & & \mathcal{M} \end{array}$$

The primary result based on this double fibration is the following:

**(B1) Non-Linear Graviton Penrose Transform (NGPT).** There is a 1–1 correspondence between anti-self-dual models $\mathscr{M} = (M, g_{ab})$ of general relativity that satisfy the vacuum Einstein equations and 4-dimensional complex manifolds $\mathcal{T}$ equipped with the following structures

(i) a four-parameter family of holomorphic curves which in $\mathbb{P}\mathcal{T}$ are compact and have normal bundle $\mathscr{O}(1) \oplus \mathscr{O}(1)$,
(ii) a projection π to primed spin space $\mathbb{S}'$,
(iii) a homogeneity operator ϒ, and
(iv) a 2-form $\tau = \varepsilon^{A'B'} d\pi_{A'} \wedge d\pi_{B'}$ and a 2-form $\mu = \varepsilon_{AB} X^{AA'} Y^{BB'} \pi_{A'} \pi_{B'}$ on each fiber over $\mathbb{S}'$.

*Comments*. For a proof, see Huggett and Tod (1994, pp. 108–109). Structure (i) corresponds to the conformal structure of $(M, g_{ab})$ while (ii), (iii), and (iv) correspond to the metric $g_{ab}$[15].

*Extensions and Limitations.* In addition to the above results, there are twistor constructions for stationary axi-symmetric vacuum solutions to the Einstein equations, extensions of *ZRMPT* for fields with sources, and extensions of Ward's Theorem for other non-linear integrable field equations (in particular, the Korteweg–de Vries equation and the non-linear Schrödinger equation). See Penrose (1999) for a review and references. Moreover, the work of Sparling (1998) demonstrates that, in principle, the twistor space corresponding to any real analytic vacuum Einstein spacetime can be constructed.

Despite these extensions, however, it should be noted that no consistent twistor descriptions have been given for massive fields or for field theories in

[15] $\mathbb{P}\mathcal{T}$ is the space of α-planes in $M$. In (i) the curves in $\mathbb{P}\mathcal{T}$ correspond to points in $M$ and the normal bundle requirement encodes the correspondence in Table 1 between null separation of points in $M$ (on which conformal structure can be based) and intersection of lines in $\mathbb{P}\mathcal{T}$. (A normal bundle $N$ to a curve γ in $\mathbb{P}\mathcal{T}$ has fibers $N_p$ consisting of all vectors at $p$ modulo tangent vectors at $p$. One can show that $N$ is a rank 2 vector bundle of the form , where $\mathscr{O}(1)$ is the sheaf of germs of homogeneous functions on $\mathbb{CP}^1$ of degree 1.) Properties (ii) and (iii) follow from the fact that $\mathbb{P}\mathcal{T}$, as the space of α-planes in $M$, becomes naturally fibered over projective spin space if $\mathscr{M}$ satisfies the vacuum Einstein equations. The 2-forms in (iv) together encode the metric $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$.

generally curved spacetimes with matter content. Essentially, as noted above, the twistor formalism is built on conformal invariance, and problems arise when it comes to rendering non-conformally invariant classical field theories. This indicates that the twistor formalism is not completely expressively equivalent to the tensor formalism, in so far as there are classical field theories that can be expressed in the latter and that cannot be expressed in the former. This may raise concerns about whether the twistor formalism should be read literally by a semantic realist. Toward assuaging these concerns, the following observations can be made.

First, to be clear, for those classical field theories outlined above, complete expressive equivalence holds between the twistor and tensor formalisms. For these examples, the twistor constructions indicate that the differentiable manifold is not essential. Second, and more importantly, while this essay is primarily concerned with classical field theories, the real (potential) benefit of the twistor formalism comes when the move is made to quantum theory. In this context, it should be noted that the verdict is still out on whether 4-dimensional interacting quantum field theories can be reformulated in a conformally invariant way. The motive for doing so stems from the fact that 2-dimensional interacting conformal field theories are exactly solvable (whereas standard formulations of 4-dimensional interacting quantum field theories are far from consistent), and from the fact that particles in any 2-dimensional quantum field theory are approximately massless in the high-energy limit (see, e.g., Gaberdiel, 2000, p. 609). Moreover, 2-dimensional conformal field theories are at the basis of string theory. (In string theory, particle masses are replaced by string tensions, and the basic Lagrangian for a propagating (bosonic) string is that of a 2-dimensional conformal field theory.) The point then is that if string theory turns out to be the correct approach to quantum gravity, for instance, or if interacting quantum field theory can be consistently reformulated in a conformally invariant way, then what tensor formulations of classical field theories have right is conformal structure, as opposed, for instance, to metrical structure predicated on points. Moreover, extensions of the twistor formalism have been proposed for formulations of 4-dimensional conformal field theory[16], and more recently, Witten (2004) has reformulated perturbative quantum Yang-Mills gauge theory as a

[16] See, e.g., Hodges, Penrose, and Singer (1989). In brief, the basic construction is referred to as a "pretzel" twistor space $P$ with boundary $\partial P$ consisting of copies of $\mathbb{PN}$. Such a space replaces the compact Riemann surface $X$ with boundary $\partial X$ that is used in 2-dimensional conformal field theory to model interacting quantum fields. $\partial X$ consists of copies of the circle $S^1$ on which complex-valued functions representing in- and out-scattering states can be defined. These functions split into negative and positive frequencies, according to whether they extend into the north or south hemispheres of the Riemann sphere with equator $S^1$. This is similar to the splitting of twistor functions defined on $\mathbb{PN}$ into negative and positive frequencies according to whether they extend into $\mathbb{PT}^+$ or $\mathbb{PT}^-$.

string theory in twistor space. The general point then is that the semantic realist should not discount twistor theory solely based on its limited applicability to classical field theories. To do so would be to ignore potential inter-theoretical relations that are key to understanding how new theories evolve from old ones[17].

### 3.2. Interpretation

How might a semantic realist take the twistor formulation of the above classical field theories at its face value? In particular, in what sense does the twistor formalism do away with the manifold of the tensor formalism? Two observations seem relevant here. First, the Penrose Transformation in all its above guises encodes the solution space of a local dynamical field equation formulated in terms of a derivative operator on a spacetime manifold, in a global geometric structure in the corresponding twistor space. In a literal sense, the local dynamics in the spacetime formulation gets encoded in a global "static" geometric structure in the twistor description, as twistor advocates like to point out.

> Note that in the Ward construction the local 'field' information in the space time description is coded in the global structure of the twistor description, whereas there is no local (differential) information in the twistor description … . This way in which local space-time field equations tend to 'evaporate' into global holomorphic structure is a characteristic (and somewhat remarkable) feature of twistor descriptions (Penrose & Rindler, 1986, p. 168).

The *dynamical* role that the manifold plays in tensor formulations of field theories is thus side-stepped in the twistor formalism; namely, the role of providing a local back-drop on which differential equations can be defined that govern the dynamical behavior of fields.

As a concrete example, tensor models of anti-self-dual CED are given by ($M$, $\eta_{ab}$, $\partial_a$, $F_{ab}$) such that

$$\eta_{ab}\partial_a F_{bc} = 0, \qquad \partial_{[a}F_{bc]} = 0, \qquad {}^{*}F_{ab} = -\mathrm{i}F_{ab} \tag{2}$$

By Ward's Theorem, twistor models of anti-self-dual *CED* may be given by ($\mathbb{PT}$, $B$), where $\mathbb{PT}$ is projective twistor space and $B$ is a line bundle over $\mathbb{PT}$ satisfying the geometrical property (A3b). Explicitly, no derivative operators occur in such twistor models.

The second observation concerns the *kinematical* role that $M$ plays in classical field theories. In the tensor formalism, traditional semantic realists have tended to read literally the mathematical fields that quantify over the points of the

[17] The twistor formalism is, in fact, generally viewed by its proponents as one route to quantum gravity. One could argue that the limitations it faces with respect to classical fields are just a particular manifestation of the obstructions to uniting quantum theory with general relativity.

manifold. The resulting literal interpretation describes physical fields that quantify over spacetime points, and that are evolved in time by means of the derivative operator associated with a connection on $M$. One might quibble over the details of such a literal interpretation: Do the manifold points really represent real substantival spacetime points? Which tensor fields defined on $M$ in the context of a given classical field theory should be awarded ontological status (potential fields vs. Yang-Mills fields, for instance)? What manifold objects should we take such fields to be quantifying over (points or loops, for instance)? But, arguably, the *nature* of the mathematical objects under debate is not in question. (Everyone agrees on what a manifold point is, for instance.) The twistor formalism is not as clear-cut.

In the twistor formalism, the mathematical tensor field has vanished, as has the derivative operator, and both have been replaced by an appropriate geometric structure defined on a twistor space. Literally, such structures quantify over the twistor space (in the same sense that tensor fields quantify over $M$). Ward's Theorem, for instance, replaces an anti-self-dual Yang-Mills field defined on $\mathbb{CM}^c$ with a vector bundle over projective twistor space $\mathbb{PT}$. Literally, this bundle is a collection of vector spaces labeled by the points of $\mathbb{PT}$, these points being projective twistors. Recall from Table 1 that, under the Klein Correspondence (*KC*), projective twistors correspond to complex null surfaces ($\alpha$-planes) in $\mathbb{CM}^c$, and when (*KC*) is restricted to real compactified Minkowski spacetime $\mathbb{M}^c$, projective twistors correspond to twisted congruences of null geodesics referred to as Robinson congruences[18]. One option, then, for a traditional semantic realist is to view such null geodesics as the individuals in the ontology of field theories formulated in the twistor formalism. Under this interpretation, twisted null geodesics are the fundamental objects, with spacetime points derivative of them (identified essentially as their intersections). This alone should give a traditional semantic realist pause for concern. But there is an additional twist: Just what the twistor individuals are is not as clear-cut as the geometric interpretation provided by the Klein correspondence might at first appear. Non-projective twistor space $\mathbb{T}$ can also be constructed *ab initio* as the phase space for a single zero rest mass particle, or as the space of charges for spin 3/2 fields (see, e.g., Penrose, 1999), or, most recently, as the space of "edge-states" for a 4-dimensional fermionic quantum Hall-effect liquid (Sparling, 2002).

To get a feel for the first of these alternative interpretations, one can show that a non-null twistor $Z^\alpha$ uniquely determines a triple $(p_a, M^{ab}, s)$, where $p_a$, $M^{ab}$ are

[18] Roughly, the real correlates of projective twistors correspond to the intersections of $\alpha$-planes and their duals, referred to as $\beta$-planes and defined with respect to the Hermitian twistor "metric" $\sum_{\alpha\beta}$. For a null twistor $Z^\alpha$ that satisfies $\sum_{\alpha\beta} Z^\alpha Z^\beta = 0$, this intersection is given by a null geodesic. For non-null twistors, the intersection is given by a Robinson congruence — a collection of null geodesics that twist about the axis defined by the null case.

tensor fields on $\mathbb{M}^c$, and $s \in \mathbb{R}$, that defines the linear momentum, angular momentum, and helicity, respectively, of a zero rest mass particle[19]. Conversely, a zero rest mass triple uniquely determines a projective twistor.

To get a feel for the second alternative interpretation, note that in Minkowski spacetime, spin-3/2 zero rest mass fields can be represented by totally symmetric spinor fields $\psi_{A'B'C'}$ (with the number of indices equal to twice the spin) that satisfy the spin-3/2 zero rest mass field equations, $\partial^{AA'}\psi_{A'B'C'} = 0$. The procedure then is to transform $\psi_{A'B'C'}$ into a spin-1 (self-dual) Maxwell field $\varphi_{A'B'}$, and then define its charge via Gauss's Law. This transformation is accomplished simply by contracting $\psi_{A'B'C'}$ on the right with a dual twistor $W_\alpha = (\lambda_A, \mu^{C'})$ to obtain $\varphi_{A'B'} = \psi_{A'B'C'}\mu^{C'}$[20]. The charge $Q$, a complex number, associated with $\psi_{A'B'C'}$ is then defined by integrating $\varphi_{A'B'}$ over a volume containing the spin-3/2 sources: $Q = \oint_S \varphi_{A'B'} \mathrm{d}S^{A'B'}$, where $S$ is the surface enclosing the sources. Since $Q$ depends linearly on $W_\alpha$, we can let $Q = Z^\alpha W_\alpha$, for some "charge" twistor $Z^\alpha$. Hence for each spin-3/2 field $\psi_{A'B'C'}$, we have a map from twistor space $\mathbb{T}$ to the space $\mathbb{C}$ of spin-3/2 charges $Q$[21].

Finally, to get a feel for the last alternative interpretation, and not get too far afield of the present essay, note that Hu and Zhang (2002) have demonstrated that the edge states of a 4-dimensional quantum Hall-effect liquid can be described by (3+1)-dimensional effective field theories of relativistic zero rest mass fields[22]. Sparling (2002) observes that their 2-spinor formalism extends naturally onto the twistor formalism and attempts to construct twistor spaces directly from Hu and Zhang's edge states.

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[19] The correspondence is given by $p_a = \bar{\pi}_A \pi_{A'}$, $M^{ab} = \mathrm{i}\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - \mathrm{i}\bar{\omega}^{(A'}\pi^{B')}\varepsilon^{AB}$, and $\sum_{\alpha\beta} Z^\alpha Z^\beta = 2s$. This ensures that the following relations that define a zero rest mass particle hold: $p_a p^a = 0$, $M^{ab} = 2r^{[a}p^{b]}$, $sp^a = 1/2\varepsilon_{abcd}\, p^b M^{cd} \equiv S^a$, where $r^a$ defines a point relative to an origin of $\mathbb{M}^c$, and $S^a$ is the Pauli–Lubanski vector.

[20] The dual twistor $W_\alpha$ is actually fully specified by $\mu^{C'}$. One can show that the (dual) twistor equation in full generality is given simply by $\partial_A{}^{(A'}\mu^{B')} = 0$. One can also show that the so-defined field $\varphi_{A'B'}$ satisfies the spin-1 zero rest mass equations $\partial^{AA'}\varphi_{A'B'} = 0$, which describe a self-dual Maxwell field.

[21] This result motivates a program in twistor theory that seeks to construct twistor spaces for full vacuum Einstein spacetimes, based on the fact that, in general, the spin-3/2 zero rest mass field equation is consistent in a spacetime $\mathscr{M}$ *if and only if* the Ricci tensor on $\mathscr{M}$ vanishes. The idea then is to look for the space of conserved charges for spin-3/2 fields on a general Ricci-flat spacetime, and this will be the corresponding twistor space.

[22] The 2-dimensional quantum Hall effect occurs when a current flowing in a 2-dimensional conductor in the presence of an external magnetic field sets up a transverse resistivity. For strong fields, this Hall resistivity is observed to be quantized in either integral or fractional units of the ratio of fundamental constants $h/e^2$. Various effective field theories have been constructed that describe this effect in terms of the properties of a highly correlated 2-dimensional quantum liquid. In particular, the low-energy excitations of the edge states of such a liquid have been described by a (1+1)-dimensional effective field theory of relativistic 2-spinor (Weyl) fields. The extension of the 2-dimensional quantum Hall effect to 4 dimensions was first given a consistent theoretical description by Zhang and Hu (2001). Their work and the similar work of others in condensed matter physics has yet to be fully considered by philosophers of spacetime.

Hence, the semantic realist committed to an individuals-based ontology has to decide between two seemingly incompatible literal construals of classical field theories: The tensor formalism suggests a commitment to local fields and spacetime points, whereas the twistor formalism suggests a commitment to twistors, which themselves admit diverse interpretations. The traditional realist might respond by claiming that the Penrose Transformation just shows that solutions to certain field equations behave in spacetime as if they were geometric/algebraic structures that quantify over twistors. In other words, we should not read the twistor formalism literally — it merely amounts to a way of encoding the behavior of the real objects, which are fields in spacetime, and which are represented more directly in the tensor formalism. In other words, we should only be semantic realists with respect to the tensor formalism. This strategy smacks a bit of *ad hocness*. All things being equal (keeping in mind the discussion at the end of Section 3.1), what, we may ask, privileges the tensor formalism over the twistor formalism? From a conventionalist's point of view, tensor fields on a manifold are just as much devices that encode the data provided by measuring devices as are vector bundles over $\mathbb{PT}$. If the semantic realist is to be genuine about her semantic realism, it appears that she must be willing to give up commitment to individuals-based ontologies and seek the basis for her literal construal at a deeper level.

## 4. Manifolds vs. Einstein algebras

In this section, I indicate how the points of a differentiable manifold can be non-trivially reconstructed from an Einstein algebra. In particular, I indicate how any classical field theory presented in the tensor formalism can be recast in the Einstein algebra formalism, and consider what this suggests about the nature of spacetime.

### 4.1. Einstein algebras and their generalizations

The Einstein algebra (*EA* hereafter) formalism takes advantage of an alternative to the standard definition of a differentiable manifold as a set of points imbued locally with topological and differentiable properties. The manifold substantivalist's gloss of this definition awards ontological status to the point set. The alternate definition emphasizes the differentiable structure, as opposed to the points of $M$ on which such structure is predicated. It is motivated by the following considerations: The set of all real-valued $C^\infty$ functions on a differentiable manifold $M$ forms a commutative ring $C^\infty(M)$ under pointwise addition and multiplication. Let $C^c(M) \subset C^\infty(M)$ be the subring of constant functions on $M$. A derivation on the pair $(C^\infty(M), C^c(M))$ is a map $X : C^\infty(M) \to C^\infty(M)$ such that $X(af+bg) = aXf+bXg$ and $X(fg) = fX(g)+X(f)g$, and $X(a) = 0$, for any $f$,

$g \in C^\infty(M)$, $a, b \in C^c(M)$. The set $\mathscr{D}(M)$ of all such derivations on $(C^\infty(M), C^c(\mathrm{M}))$ forms a module over $C^\infty(M)$ and can be identified with the set of smooth contravariant vector fields on $M$. A metric $g$ can now be defined as an isomorphism between the module $\mathscr{D}(M)$ and its dual $\mathscr{D}^*(M)$. Tensor fields may be defined as multi-linear maps on copies of $\mathscr{D}(M)$ and $\mathscr{D}^*(M)$, and a covariant derivative can be defined with its associated Riemann tensor. Thus all the essential objects of the tensor formalism necessary to construct a model of general relativity (*GR*) may be constructed from a series of purely algebraic definitions based ultimately on the ring $C^\infty(M)$. At this point Geroch's (1972) observation is that the manifold only appears initially in the definition of $C^\infty(M)$. This suggests viewing $C^\infty$ and $C^c$ as algebraic structures in their own right, with $M$ as simply a point set that induces a representation of them[23]. Formally, Geroch (1972) defined an Einstein algebra $\mathscr{A}$ as a tuple $(\mathscr{R}^\infty, \mathscr{R}, g)$, where $\mathscr{R}^\infty$ is a commutative ring, $\mathscr{R}$ is a subring of $\mathscr{R}^\infty$ isomorphic with the real numbers, and $g$ is an isomorphism from the space of derivations on $(\mathscr{R}^\infty, \mathscr{R})$ to its algebraic dual such that the associated Ricci tensor vanishes (and a contraction property is satisfied)[24].

Two observations are relevant at this point. First, Geroch's algebraic treatment of *GR* can be trivially generalized to include all classical field theories presented in the tensor formalism. In general, the latter are given by tuples $(M, O_i)$, where $M$ is a differentiable manifold and the $O_i$ are tensor fields defined on $M$ and satisfying the appropriate field equations (via a derivative operator on $M$). After Earman (1989), let a Leibniz algebra $\mathscr{L}$ be a tuple $(\mathscr{R}^\infty, \mathscr{R}, A_i)$, where $\mathscr{R}^\infty$ is a commutative ring, $\mathscr{R}$ is a subring isomorphic with the real numbers, and the $A_i$ are algebraic objects defined as multi-linear maps on copies of $\mathscr{D}$ (the set of all derivations on $(\mathscr{R}^\infty, \mathscr{R})$) and its dual $\mathscr{D}^*$, and satisfying a set of field equations (via the algebraic correlate of a derivative operator). For an appropriate choice of $A_i$, such an $\mathscr{L}$ is the correlate in the *EA* formalism of a model of a classical field theory in the tensor formalism.

The second observation concerns the extent to which an Einstein (or Leibniz) algebra is expressively equivalent to a tensor model of a classical field theory. In particular, in what sense is the manifold $M$ done away with in the *EA* formalism? There seems to be both a trivial and a non-trivial sense in which $M$ is done away with. The trivial sense is based on the following considerations. The maximal ideals of an abstract algebra $\mathscr{A}$ (if they exist) are in 1–1 correspondence with the

[23]Such a representation is given by the Gelfand representation. Any abstract linear algebra $\mathscr{A}$ (over a field ) admits a Gelfand representation defined by $\rho : \mathscr{A} \to \mathbb{K}^{\mathscr{A}^*}, \rho(x)(\phi) = \phi(x)$, where $x \in \mathscr{A}$, $\phi \in \mathscr{A}^*$, and $\mathscr{A}^*$ is the algebraic dual of $\mathscr{A}$ (i.e., the set of homomorphisms $\phi: \mathscr{A} \to \mathbb{K}$) and $\mathbb{K}^{\mathscr{A}^*}$ is the algebra of $\mathbb{K}$-valued functions on $\mathscr{A}^*$. Intuitively, the Gelfand representation turns the abstract object $\mathscr{A}$ into a "concrete" algebra of functionals on a space $\mathscr{A}^*$.

[24]The above deviates slightly from Geroch's notation. The condition on the algebraic Ricci tensor can be relaxed and algebraic correlates of the Einstein tensor and cosmological constant can be introduced to model general solutions to the Einstein equations (see, e.g., Heller, 1992).

elements of its algebraic dual $\mathscr{A}^*$[25]. Hence, if $\mathscr{A}$ has maximal ideals, the points of the space $\mathscr{A}^*$ can be reconstructed by means of the Gelfand representation of $\mathscr{A}$ (see footnote 23). In particular, the points of a topological space $X$ can be reconstructed from the maximal ideals of the ring $C(X)$. (Concretely, one shows that any maximal ideal of $C(X)$ consists of all functions that vanish at a given point of $X$.) A differentiable manifold $M$ can then be reconstructed by imposing a differentiable structure (i.e., a maximal atlas) on $X$[26]. Hence, there is a 1–1 correspondence between Einstein (Leibniz) algebras and models of classical field theories in the tensor formalism, *and this correspondence extends all the way down to the point set of M*. This suggests that, from the point of view of literal interpretations of spacetime, nothing is gained in moving to the *EA* formalism: any interpretive options under consideration in the tensor formalism will be translatable in 1–1 fashion into the *EA* formalism[27].

The non-trivial sense will have to wait until the next section, after some extensions of the *EA* formalism have been reviewed.

*Extensions.* Heller and Sasin have extended Geroch's original treatment of *GR* to spacetimes with singularities. A non-singular general relativistic spacetime can be represented by a differentiable manifold $M$, or an Einstein algebra generated by the ring $C^\infty(M)$. To represent certain types of curvature singularities in the tensor formalism requires additional structures on $M$. In particular, the $b$-boundary construction collects singularities in a space $\partial_b M$ and attaches it as a boundary to $M$ to create a differentiable manifold with boundary $M' = M \cup \partial_b M$. In the *EA* formalism, one can now consider an algebraic object of the schematic form $C^\infty(M')$, consisting of real-valued $C^\infty$ functions on $M'$. Originally, this object was identified as a sheaf of (commutative) Einstein algebras over $M'$ (Heller & Sasin, 1995). Heller and Sasin (1996) demonstrated that such an object can also be analyzed as a non-commutative Einstein algebra of complex-valued $C^\infty$ functions over a more general structure (in particular, the semi-direct product $OM \rtimes O(1, 3)$, of the Cauchy completed frame bundle $OM$ over $M'$ and the structure group $O(1, 3)$). This analysis was then extended to a schema for quantum gravity in Heller and Sasin (1999). The theory presented there takes as the fundamental object an "Einstein $C^*$-algebra" $\mathscr{E}$, constructed

[25]Elements of $\mathscr{A}^*$ are sometimes called the "characters" of $\mathscr{A}$. A maximal ideal of $\mathscr{A}$ is the largest proper subset of $\mathscr{A}$ closed under (left or right) multiplication by any element of $\mathscr{A}$.

[26]Note that there are (at least) two ways to view the reconstruction of points of a differentiable manifold. One can reconstruct the points of a topological space $X$ from the maximal ideals of $C(X)$, and then impose a differentiable structure on $X$ to obtain a differentiable manifold. Alternatively, one can directly reconstruct the points of $M$ from the maximal ideals of $C^\infty(M)$. See, e.g., Demaret, Heller, and Lambert (1997, p. 163).

[27]In particular, some authors have claimed interpretive issues surrounding the hole argument cannot be addressed simply by moving to the *EA* formalism. For a discussion, see Bain (2003).

from the non-commutative algebra of complex-valued $C^{\infty}$ functions with compact support on a transformation groupoid (see Bain, 2003 for a brief review).

## 4.2. Interpretation

As indicated above, there is a trivial sense in which the original *EA* formalism does away with manifolds; namely, simply by renaming them: instead of manifold points, the original *EA* talk is about maximal ideals. One might argue that renaming an object does not make it go away. In particular, Einstein algebras for non-singular spacetimes reproduce the diffeomorphism "redundancy" of *M*. An argument could be made, however, that the extended *EA* formalism does do away with *M* in a non-trivial manner. First, as Heller and Sasin (1995) note, the (commutative) extensions of *EA* to singular spacetimes in effect place non-singular and singular spacetimes under a single category (namely, the category of "structured spaces": spaces structured by a sheaf of Einstein algebras); whereas in the tensor formalism, technically, non-singular and singular spacetimes belong to different categories (the categories of smooth manifolds and manifolds with boundaries, respectively)[28]. In not talking about manifold points to begin with, the extended *EA* formalism can handle field theories characterized by missing manifold points in a conceptually cleaner manner than the tensor formalism.

Heller and Sasin (1995) further suggest that certain conceptual problems associated with the *b*-boundary construction in the tensor formalism do not arise in the extended *EA* formalism. Briefly, in the closed Friedman universe (of Big Bang fame), the *b*-boundary consists of a single point corresponding to both the initial and final singularities, and in both the closed Friedman and Schwarzschild solutions, the *b*-boundary is not Hausdorff-separated from *M*. These results are hard to reconcile with any notion of localization. (Intuitively, some amount of separation between the initial and final singularities in the Friedman solution should obtain.) Moreover, that the points of the *b*-boundary are not Hausdorff-separated from the points of the interior implies counter intuitively that every event in spacetime is in the neighborhood of a singularity. The suggestion of Heller and Sasin (1995) is that these decidedly non-local aspects of *b*-boundary constructions are pathologies only when viewed from within the differentiable manifold category and its emphasis on local properties. In the extended *EA* formalism (in particular, in the category of structured spaces), in contrast, the emphasis throughout is on sheaf-theoretic global features, and these features allow a natural distinction between the decidedly non-local behavior of fields on the *b*-boundary and the local behavior of fields on the interior *M*.

[28]Unlike a manifold with boundary, a smooth ($C^{\infty}$) differentiable manifold is differentiable at all points; intuitively, it has no "edge points" at which differentiation may break down. For the theory of structured spaces, see Heller and Sasin (1995) and references therein.

A second point is that in the non-commutative extensions of *EA* given in Heller and Sasin (1996, 1999), the manifold *M* truly disappears. In these extensions, a commutative algebra is replaced with a non-commutative algebra, and, simply put, these latter, in general, have no maximal ideals. Thus well-behaved point sets cannot, in general, be reconstructed from them. Intuitively, one might claim that Einstein algebras, both commutative and non-commutative, encode the *differentiable structure* of a differentiable manifold first and foremost, and only secondarily encode *M*'s point set.

How might a semantic realist take the *EA* formulation of classical field theories at its face value? In particular, what might a literal interpretation of a (commutative) Einstein algebra amount to? In the original *EA* formalism, the correlates of manifold points are the maximal ideals of the algebra $\mathscr{A}$. Under the Gelfand representation, these are certain subsets of functionals defined on $\mathscr{A}^*$, which, under the intended manifold interpretation, become real-valued $C^\infty$ functions defined on *M*. Some authors have suggested that these functions can be interpreted as a system of scalar fields, which the literal-minded semantic realist can include in her ontology *in lieu* of manifold points (see, e.g., Penrose & Rindler, 1984, p. 180; Demaret et al., 1997, p. 146). This interpretation suggests a notion of spacetime as arising out of the relations between these fundamental fields[29].

In the extended *EA* formalism, we have replaced commutative algebras with non-commutative algebras, and these latter, in general, do not possess maximal ideals. Hence, there are, in general, no correlates of manifold points to help the literal-minded semantic realist. One option for the semantic realist is a literal interpretation not of the objects of any particular representation of an Einstein algebra (commutative or not), but rather of the algebraic structure intrinsic to the algebra itself. An Einstein algebra $\mathscr{A}$ can be realized in many ways on many different types of spaces. Some of these spaces can be interpreted as smooth differentiable manifolds, others as manifolds with boundaries, and still others do not admit a manifold interpretation at all. An "algebraic structuralist" might claim that the concrete representations of $\mathscr{A}$ should not be read literally; rather, the structure defined by the algebraic properties of $\mathscr{A}$ is what should be taken at face value.

## 5. Manifolds vs. geometric algebra

In this section, I indicate how classical field theories can be recast using geometric algebra and the extent to which the geometric algebra formalism is

[29]Relationalists like Rovelli (1997) hold a similar view with respect to the metric field in tensor formulations of general relativity. Note, however, that such metric field relationalists differ from algebraic relationalists in so far as the former posit a single "manifold-generating" field that has physical significance (being a solution to the Einstein equations), whereas the latter require an uncountable infinity of fields, most of which will not have physical significance. (Thanks to an anonymous referee for making this point explicit.)

non-trivially expressively equivalent to the tensor formalism. Whereas an Einstein algebra may be said to encode the differentiable structure of a manifold in an abstract algebraic object, a geometric algebra on first glance may be said to encode the *metrical structure* of a manifold in a *concrete* algebra of "multivectors". As it turns out, there is also an abstract algebraic object lurking behind the scenes here, too; namely, an abstract Clifford algebra.

In slightly more detail, a geometric algebra $\mathcal{G}$ can be initially viewed as a generalization of a vector space. The elements of $\mathcal{G}$ are referred to as multivectors and come in "grades". The intended geometrical interpretation identifies 0-grade multivectors as scalars, 1st-grade multivectors ("1-vectors") as vectors, 2nd-grade multivectors ("bivectors") as directed surfaces, 3rd-grade multivectors ("trivectors") as directed volumes, etc. For any $r$, the collection of all $r$-grade multivectors forms a subalgebra $\mathcal{G}^r$ of $\mathcal{G}$, with $\mathcal{G}$ then being the direct sum of all the $\mathcal{G}^r$, $r = 0 \ldots \infty$. This allows any $n$-dimensional vector space $V^n$ to be identified with a geometric algebra $\mathcal{G}(V^n)$ for which $\mathcal{G}^1(V^n) = V^n$. The real significance of $\mathcal{G}$ lies in the geometric product which encodes both an inner product (bilinear form) and an outer (wedge) product. These properties of $\mathcal{G}$ allow classical field theories to be presented in the geometric algebra (*GA* hereafter) formalism in an intrinsically coordinate free manner in a way that does away with the differentiable manifold of the tensor formalism.

### 5.1. Geometric algebra

From a mathematical point of view, a geometric algebra $\mathcal{G}$ is first and foremost a real Clifford algebra. There are numerous ways of defining the latter. For instance, let $V$ be a real vector space equipped with a bilinear form $g : V \times V \to \mathbb{R}$ with signature $(p, q)$. The real Clifford algebra $\mathcal{C}_{(p,q)}$ is the linear algebra over $\mathbb{R}$ generated by the elements of $V$ via "Clifford multiplication" defined by $xy + yx = g(x, y)\mathbf{1}$, $x, y \in V$, where $\mathbf{1}$ is the unit element. In this, and other standard definitions, a Clifford algebra is defined in terms of a bilinear form (or its associated quadratic form) defined on a vector space[30]. Given such definitions, Clifford algebras might seem limited to applications in metrical geometry, or might seem less fundamental than tensor algebra. The axiomatic treatment of Hestenes and Sobczyk (1984) is meant to address these apparent limitations.

[30] An alternative definition is the following (Ward & Wells, 1990, p. 209): Let $V$ be a vector space over a commutative field $\mathbb{K}$ with unit element $\mathbf{1}$ and equipped with a quadratic form $q : V \to \mathbb{K}$. (Such a $q$ is defined by $q(xr) = r^2 q(x), r \in \mathbb{K}, x \in V$ such that the map $h : V \times V \to \mathbb{K}$ defined by $h(x, y) = q(x+y) - q(x) - q(y)$ is a bilinear form on $V$. A simple consequence of this definition is that $h(x, x) = 2q(x)$.) The tensor algebra of $V$ is given by $\mathcal{T}(V) = \Sigma_{r=0}^{\infty} \otimes^r V$. Let $\mathcal{J}$ be the two-sided ideal in $\mathcal{T}(V)$ generated by elements of the form $x \otimes x + q(x)\mathbf{1}$, for $x \in V$. The Clifford algebra associated with $V$ is then defined as the quotient $\mathcal{C}(V, q) \equiv \mathcal{T}(V)/\mathcal{J}$. The Clifford product in $\mathcal{C}(V, q)$ is then the product induced by the tensor product in $\mathcal{T}(V)$.

Their goal is to construct a real Clifford algebra (now referred to as a geometric algebra) as a primitive object in its own right, with the notions of vector space and bilinear form as derivative concepts. In what follows, I will briefly review their axiomatic construction before reviewing its application to classical field theories in Minkowski spacetime and in generally curved spacetimes.

In Hestenes and Sobczyk's (1984) treatment, a geometric algebra $\mathcal{G}$ is a *graded real associative* algebra with a few additional properties. Elements of $\mathcal{G}$ are referred to as multivectors. As a *real associative* algebra, a geometric algebra is a septuple $\mathcal{G} = (\mathcal{G}, +_g, \times_g; \mathbb{R}, +, \times; *)$, where $(\mathcal{G}, +_g, \times_g)$ is a ring with unity closed under geometric addition $+_g$ and non-commutative geometric multiplication $\times_g$; $(\mathbb{R}, +, \times)$ is the real field, and $*$ denotes the external binary operation of scalar multiplication (in the following, the subscript on $+_g$ has been dropped and $\times_g$ and $*$ are represented by juxtaposition). As a *graded* algebra, $\mathcal{G}$ admits a linear idempotent grade operator $\langle\ \rangle_r$: $\mathcal{G} \to \mathcal{G}$ by means of which any multi-vector $A \in \mathcal{G}$ can be written as the sum $A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots = \sum_r \langle A\rangle_r$. If $A = \langle A\rangle_r$, then $A$ is referred to as homogeneous of grade $r$ and called an *r-vector*. The space of all $r$-vectors is denoted $\mathcal{G}^r$ and is an $r$-dimensional linear subspace of $\mathcal{G}$. The space $\mathcal{G}^0$ is identified with $\mathbb{R}$. The role of the bilinear form in standard treatments is accomplished by including an axiom relating scalar and vector multiplication: for $a \in \mathcal{G}'$, $aa = a^2 = \langle a^2\rangle_0$. In words: the square (under geometric multiplication) of a "1-vector" is a scalar[31]. This relation is then extended to arbitrary $r$-vectors by the axiom: For any $r > 0$, an $r$-vector can be expressed as a sum of *r-blades*, where $A_r$ is an $r$-blade *iff* $A_r = a_1 a_2 \ldots a_r$, where $a_j a_k = -a_k a_j$, for $j, k = 1 \ldots r$ and $j \neq k$. Finally, Hestenes and Sobczyk posit the existence of non-trivial blades of every finite grade: For every non zero $r$-blade $A_r$, there exists a non-zero vector $a$ in $\mathcal{G}$ such that $A_r a$ is an $(r+1)$-blade. (Hence, $\mathcal{G}$ is infinite dimensional.)

The geometric product can be decomposed into an inner product and an outer product. For homogeneous multivectors, the inner product $\cdot$ and the outer product $\wedge$ are defined by $A_r \cdot B_s \equiv \langle A_r B_s\rangle_{|r-s|}$, if $r$ and $s > 0$, otherwise $A_r \cdot B_s \equiv 0$, and $A_r \wedge B_s \equiv \langle A_r B_s\rangle_{r+0}$[32]. Intuitively, the inner product decreases the grade of multivectors, whereas the outer product increases grade. These definitions entail that the geometric product of a 1-vector $a$ and an arbitrary multivector $A$ can be decomposed as $aA = a \cdot A + a \wedge A$. In particular, the geometric product of 1-vectors has the simple decomposition $ab = a \cdot b + a \wedge b$, where $a \cdot b = 1/2(ab + ba)$

[31]In standard treatments, $\mathcal{G}$ would be identified as the Clifford algebra of the quadratic form $q(a) = a^2$ with associated bilinear form $h(a, b) = (a+b)^2 - a^2 - b^2$ (see previous footnote).

[32]For arbitrary multivectors, they are defined as $A \cdot B \equiv \sum_r \sum_s \langle A\rangle_r \cdot \langle B\rangle_s$ and $A \wedge B \equiv \sum_r \sum_s \langle A\rangle_r \wedge \langle B\rangle_s$.

is the totally symmetric part of $ab$, and $a \wedge b = 1/2(ab - ba)$ is the totally anti-symmetric part of $ab$[33].

Every $n$-dimensional vector space $V^n$ determines a subalgebra $\mathcal{G}(V^n)$ of $\mathcal{G}$ by geometric multiplication and addition of elements in $V^n$ such that $\mathcal{G}^1(V^n) = V^n$ and $\mathcal{G}^r(V^n)$ is the linear subspace of $\mathcal{G}(V^n)$ consisting of all $r$-vectors formed by taking products of elements of $V^n$. In particular, let $\{e_1, \ldots, e_n\}$ be a basis for $V^n$. Then a basis for $\mathcal{G}^r(V^n)$ is given by $\{1, e_i, e_{i_1} e_{i_2}, \ldots, e_{i_1} \ldots e_{i_n}\}$, $i = 1 \ldots n$, and a multivector element $B \in \mathcal{G}^r(V^n)$ may be expanded as, $B = c + c^i e_i + c^{i_1 i_2} e_{i_1} e_{i_2} + \cdots + c^{i_1 \cdots i_n} e_{i_1} \ldots e_{i_n}$, where the $c^i$ are scalar coefficients. $\mathcal{G}(V^n)$ can thus be decomposed into a direct sum of linear subspaces $\mathcal{G}^r(V^n)$. Note that the dimension of $\mathcal{G}(V^n)$ is $2^n$.

Two subalgebras of $\mathcal{G}$ play essential roles in the formulation of classical field theories in the *GA* formalism: the Pauli algebra associated with Euclidean 3-space and the Dirac algebra associated with Minkowski spacetime.

*Pauli algebra and Dirac algebra*. The Pauli algebra $\mathcal{P}$ is the geometric algebra $\mathcal{G}(E^3)$ (alternatively, the real Clifford algebra $\mathcal{C}_{(0,3)}$) of the vector space $E^3$ tangent to a point in Euclidean 3-space. A basis for $E^3$ is given by $\{\sigma_1, \sigma_2, \sigma_3\}$, where the basis 1-vectors satisfy $\sigma_I \cdot \sigma_j = \delta_{ij}$, $\sigma_i \wedge \sigma_j = 0$[34]. The corresponding 8-dimensional basis for $\mathcal{P}$ is then,

$$\{1,\ \{\sigma_1, \sigma_2, \sigma_3\},\quad \{\sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3\},\quad \{\sigma_1\sigma_2\sigma_3\}\} \tag{3}$$

where, e.g., $\sigma_1\sigma_2 = \sigma_1 \cdot \sigma_2 + \sigma_1 \wedge \sigma_2 = -\sigma_2\sigma_1$. Note that the highest-grade basis element (or "pseudoscalar") $\sigma_1\sigma_2\sigma_3$ of $\mathcal{P}$ has the properties $(\sigma_1\sigma_2\sigma_3)^2 = -1$ and $(\sigma_1\sigma_2\sigma_3)\sigma_\mathrm{k} = \sigma_\mathrm{k}(\sigma_1\sigma_2\sigma_3)$, i.e., $\sigma_1\sigma_2\sigma_3$ commutes with all basis elements. This motivates the denotation $\sigma_1\sigma_2\sigma_3 \equiv i$. Hereafter, "$i$" will denote the pseudoscalar of $\mathcal{P}$ (and, as will be seen, that of the Dirac algebra $\mathcal{D}$ as well). Any $A \in \mathcal{P}$ can be expanded in the basis (3) as $A = \alpha + \mathbf{a} + i\mathbf{b} + i\beta$, where $\mathbf{a} = a_k\sigma_k$, $\mathbf{b} = b_k\sigma_k$ are 1-vector elements of $\mathcal{P}^1 \equiv \mathcal{G}^1(E^3)$ and $a_k$, $b_k$, $\alpha$, $\beta$ are scalars.

The Dirac, or spacetime, algebra $\mathcal{D}$ is the geometric algebra $\mathcal{G}(M^4)$ of Minkowski vector space $M^4$ (alternatively, the real Clifford algebra $\mathcal{C}_{(1,3)}$). It can be generated by the set of 1-vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, satisfying $\gamma_0\gamma_0 = 1$, $\gamma_k\gamma_k = -1$, and $\gamma_\mu \cdot \gamma_v = 0$ for $\mu \neq v$[35]. The Minkowski metric $\eta_{\mu v}$ is then recovered as

[33] In standard treatments, the inner product is defined by the bilinear form $h(x, y) = x \cdot y$ associated with the quadratic form $q(x) = x^2$. The outer product is the wedge product of tensor algebra.

[34] The Pauli operator algebra of non-relativistic quantum mechanics can be realized in $\mathcal{P}$ (hence the name). Under this realization, the 1-vectors $\sigma_1$, $\sigma_2$, $\sigma_3$ correspond to the Pauli spin matrix operators, and 2-component $SU(2)$ "non-relativistic" spinors correspond to even elements of $\mathcal{P}$ (see, e.g., Lasenby, Doran, & Gull, 1993). Thus, insofar as $\mathcal{P}$ is a real Clifford algebra in which the object $i$ has a definite geometric interpretation (see below), one can reconstruct the kinematics of non-relativistic quantum mechanics in $\mathcal{P}$ without introducing the complex field $\mathbb{C}$.

[35] The Dirac operator algebra of relativistic quantum mechanics can be realized in $\mathcal{D}$ (hence the name). Under this realization, the 1-vectors $\gamma_\mu$ correspond to the Dirac matrix operators, and 4-component Dirac spinors correspond to even elements of $\mathcal{D}$ (see, e.g., Lasenby et al., 1993).

$\gamma_\mu \cdot \gamma_v = 1/2(\gamma_\mu \gamma_v + \gamma_v \gamma_\mu) = \eta_{\mu v}$ The corresponding 16-dimensional basis for $\mathscr{D}$ is given by

$$\{1,\ \{\gamma_\mu\},\ \{\sigma_k, \mathrm{i}\sigma_k\},\ \{\mathrm{i}\gamma_\mu\},\ i\} \tag{4}$$

where the pseudoscalar of $\mathscr{D}$ is given by $\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3 = i$, and $\sigma_k \equiv \gamma_k\gamma_0$, $k = 1 \ldots 3$, are bivectors (in $\mathscr{D}$) that form an orthonormal frame in the Euclidean 3-space orthogonal to the $\gamma_0$ direction. In terms of this basis, the Pauli algebra generated by the $\sigma_k$ is the even subalgebra of $\mathscr{D}$. Vectors in $\mathscr{D}$ are embedded into $\mathscr{P}$ by geometric right-multiplication by $\gamma_0$, bivectors in $\mathscr{D}$ are embedded into $\mathscr{P}$ by expansion in the basis $\{\sigma_k, i\sigma_k\}$, and scalars and pseudoscalars in $\mathscr{D}$ remain scalars and pseudoscalars in $\mathscr{P}$.

Any $A \in \mathscr{D}$ can be expanded in the basis (4) as $A = A_S + A_V + A_B + A_T + A_P$, where the labels $S$, $V$, $B$, $T$, $P$ refer to the scalar, vector, bivector, trivector, and pseudoscalar part of $A$, respectively. Geometric interpretations of these objects are as follows: Scalars are elements of the subalgebra $\mathscr{G}^0(M^4)$, identified with $\mathbb{R}$; elements of the subalgebra $\mathscr{G}^1(M^4) = M^4$ are Minkowski 4-vectors; elements of the subalgebra $\mathscr{G}^2(M^4)$ are bivectors: directed surface elements in $M^4$; elements of the subalgebra $\mathscr{G}^3(M^4)$ are trivectors: directed volume elements in $M^4$; and elements of $\mathscr{G}^4(M^4)$ are pseudoscalars: directed hypervolumes in $M^4$.

*Fields and derivative operators*. Physical fields are represented in the *GA* formalism by geometric functions. A *geometric* function $F(A)$ is a function whose domain and range are subsets of $\mathscr{G}$. The standard definitions of limit and continuity for scalar-valued functions on $\mathbb{R}^n$ can now be employed for geometric functions using the scalar magnitude, which defines a unique distance $|A - B|$ between any two multivectors $A$, $B$[36]. A geometric function of $r$ variables $T = T(A_1, A_2, \ldots, A_r)$ is called an *extensor of degree r on* $\mathscr{G}^n$ if it is linear in each of its arguments and each variable is defined on a geometric algebra $\mathscr{G}^n$. In particular, if $n = 1$, then $T$ is a *tensor of degree* $r$. A tensor $T = T(a_1, \ldots, a_r)$ of degree $r$ that takes values in a geometric algebra $\mathscr{G}^s$ is said to have grade $s$ and rank $s + r$.

A geometric calculus for the Dirac algebra $\mathscr{D}$ can be constructed by extending the well-defined notion of derivative in $\mathscr{G}^0(M^4)$ to all of $\mathscr{D}$. Naively, this is possible since both addition and multiplication are well defined for all elements of $\mathscr{D}$ (hence, specifically, limits of quotients can be defined). In general, the vector derivative $\partial$ for $\mathscr{G}$ is defined by $\partial \equiv e^\mu(e_\mu \cdot \partial_x)$ where $\{e_\mu\}$ is a basis for $\mathscr{G}^1$ and $(e_\mu \cdot \partial_x)$ is a scalar derivative operator[37]. The vector derivative $\partial$ so-defined

[36] For arbitrary $A, B \in \mathscr{G}$, the *scalar product* $*$ is defined as $A * B \equiv \langle AB \rangle_0$ (note that this is distinct from the inner product). The *scalar magnitude* of $A$ is then defined by $|A|^2 \equiv A^\dagger * A$, where the *reversion map* $\dagger$: $\mathscr{G} \to \mathscr{G}$ is defined on $r$-vectors by $A_r{}^\dagger = (-1)^{r(r-1)/2} A_r$ (reversion reverses the order of all products of 1-vectors in $A_r$).

[37] In general, let $F(x)$ be a multivector-valued geometric function of $x \in \mathscr{G}^1$ on $\mathscr{G}$ and let $a \in \mathscr{G}^1$. The *directional derivative* of $F(x)$ in the direction of $a$ is defined by $(a \cdot \partial_x)F(x) \equiv \lim_{\tau \to 0}(F(x + \tau a) - F(x))/\tau$ One can show that the operator $(a \cdot \partial_x)$ has all the properties of a scalar derivative operator (Hestenes & Sobczyk, 1984, pp. 44–53).

is the geometric product of a 1-vector $e^{\mu}$ and a scalar differential operator $(e_{\mu} \cdot \partial_x)$, acquiring the algebraic properties of a 1-vector from the former and differential properties from the latter. Since it is a vector quantity its action on geometric functions can be decomposed into inner and outer products: For any differentiable geometric function $A(x)$ of a vector argument with values in $\mathcal{G}$, $\partial A(x) = \partial \cdot A(x) + \partial \wedge A(x)$. To specialize to $\mathcal{D}$, let $\{\gamma_{\mu}\}$ be a basis for $\mathcal{D}^1$. Then the vector derivative for $\mathcal{D}$ is given by $\partial \equiv \gamma^{\mu}(\gamma_{\mu} \cdot \partial_{\mu})$, where $\{\gamma^{\mu}\}$ is the reciprocal basis defined by $\gamma^{\mu} \cdot \gamma_{\nu} = \eta^{\mu}{}_{\nu}$.

We are now in the position of being able to transcribe classical field theories in Minkowski spacetime into the *GA* formalism. In all such transcriptions, the differentiable manifold *M* that appears in the tensor formalism is replaced with the Dirac algebra $\mathcal{D}$. As an example, the Maxwell equations can be written in the *GA* formalism as

$$\partial F = 4\pi J \tag{5a}$$

where the electromagnetic field $F = F(x)$ is a bivector-valued tensor on $\mathcal{D}^1$ (i.e., a tensor of degree 1, grade 2, and rank 3) and the current density $J = J(x)$ is a tensor on $\mathcal{D}^1$ of degree 1, grade 1 and rank 2. To show that (5a) reproduces the Maxwell equations, it can be decomposed into

$$\partial \cdot F = 4\pi J, \qquad \partial \wedge F = 0 \tag{5b}$$

These equations then reproduce the standard tensor formulation (1) in a given basis $\{\gamma_{\mu}\}$.

To formulate general relativity in the *GA* formalism, two options are available. First, Lasenby, Doran, and Gull (1998) have constructed a gauge theory of gravity in flat Minkowski vector space that reproduces the Einstein equations and that is similar to Poincaré gauge theory formulations of *GR*. In these latter theories, one typically imposes local Poincaré gauge invariance on a matter Lagrangian, which requires the introduction of gauge potential fields. These are then identified as the connection (rotational gauge) on a Poincaré frame bundle over a manifold *M*, and the tetrad fields (translation gauge). The Einstein equations are then obtained by extremizing the Lagrangian with respect to the gauge potentials. In Lasenby et al. (1998), "displacement" and rotational gauge invariance is imposed on a matter Lagrangian defined on the Dirac algebra $\mathcal{D}$, and this leads to the introduction of potential gauge fields defined on $\mathcal{D}$ that generate the Einstein equations (plus an equation for torsion). In this theory, gravity is conceived as a force described by geometric functions defined on the Dirac algebra.

The second option is to attempt to transcribe *GR* as a theory governing fields in a curved spacetime directly into the *GA* formalism. To accomplish this, one can make use of Hestenes and Sobczyk's (1984, Chapter 4) notion of a *vector manifold*: a collection of 1-vector elements of $\mathcal{G}$. A vector manifold $\mathcal{M}$ can be considered as a curved surface embedded in a larger flat space (associated with $\mathscr{G}$). The extrinsic geometry of $\mathscr{M}$ can be defined in terms of objects in the "embedding space" $\mathscr{G}$, and an intrinsic (Riemannian) geometry can be defined in $\mathscr{M}$ by projecting the relevant quantities in $\mathscr{G}$ onto $\mathscr{M}$. In particular, a curvature tensor can be defined as a geometric function on $\mathscr{M}$ and this then allows the transcription of the Einstein equations as equations governing geometric function fields defined on $\mathscr{M}$[38].

## 5.2. *Interpretation*

In what sense does the *GA* formalism do away with the manifold *M* of the tensor formalism? Note that, for classical field theories in Minkowski spacetime, including the *GA* gauge theory of gravity of Lasenby et al. (1998), the *kinematical* role of *M* as a point-set for tensor fields to quantify over is explicitly played by the subalgebra $\mathscr{D}^1$ of 1-vector elements of the Dirac algebra $\mathscr{D}$, in so far as physical fields in the *GA* formalism are represented by geometric tensor functions that quantify over 1-vectors. The *dynamical* role of *M* as a set of points imbued with differentiable and topological properties on which derivative operators may be defined is also played by $\mathscr{D}^1$ with its associated vector derivative $\partial$. On the other hand, a case could be made that the object in the *GA* formalism that plays both the kinematical and dynamical roles of *M* is the Dirac algebra $\mathscr{D}$ in its entirety. Recall that $\mathscr{D}$ is the direct sum $\mathscr{D}^0(=\mathbb{R})+\mathscr{D}^1+\mathscr{D}^2+\mathscr{D}^3+\mathscr{D}^4$. Geometric tensor functions in $\mathscr{D}$ quantify over $\mathscr{D}^1$ and take values in any of these subalgebras of $\mathscr{D}$. Hence physical fields, in this sense, are represented simply by elements of $\mathscr{D}$. Moreover, the vector derivative operator $\partial \in \mathscr{D}^1$ is only well defined as a derivative operator due essentially to the differentiable properties of $\mathscr{D}^0$[39]. The claim then is that $\mathscr{D}$ comes as a self-contained package: to use any one aspect of it in formulating a classical field theory in Minkowski spacetime requires making use of $\mathscr{D}$ in its entirety. (Arguably, this is not the case in the tensor formalism in which *M* is considered as a "self-contained" mathematical object in its own right with additional structures defined on it as the need arises.)

[38]As Doran, Lasenby, and Gull (1993) note, one drawback of this approach is that the Einstein equations in their tensorial form only determine the local curvature of *M* and, in general, say nothing about its global properties. In contrast, a vector manifold, as an embedded surface, has a well-defined global extrinsic curvature. Hence to fully accommodate vector manifolds into *GR*, the Einstein equations should be modified to specify such extrinsic properties. Furthermore, there are topological issues associated with both options of incorporating *GR* into the *GA* formalism, due to the well-behaved (topologically) features of vector spaces *vis-á-vis* differentiable manifolds.

[39]In addition to a vector ($\mathscr{D}^1$) derivative, higher-grade derivatives associated with each of the other subalgebras of $\mathscr{D}$ can be defined. The general theory of such multivector derivatives is presented in Hestenes and Sobczyk (1984, p. 54).

To make this a bit more explicit, consider, once again, *CED* in Minkowski spacetime. In the *GA* formalism, a dynamical model for *CED* in Minkowski spacetime may be given by ($\mathscr{D}$, $\partial$, $F$, $J$), where $\mathscr{D} \subset \mathscr{G}$ is the Dirac algebra, $\partial$ is the vector derivative of $\mathscr{D}$, and the electromagnetic bivector $F \in \mathscr{D}^2$ and the current density vector $J \in \mathscr{D}^1$ satisfy the *GA* formulation of the Maxwell equations (5a). Here, the Dirac algebra in its entirety replaces ($M$, $\eta_{ab}$) as the object encoding the properties of spacetime.

How might a semantic realist take the *GA* formulation of classical fields at its face value? Unlike the Einstein algebra case, *GA* comes pre-packaged with an intended interpretation. The objects of a geometric algebra, and the Dirac algebra in particular, are interpreted as multivectors. One option for a semantic realist is to include them as the fundamental geometric entities in the ontology of classical field theories. This perhaps suggests a relationalist's view of spacetime as arising from the algebraic relations between multivectors in the Dirac algebra. Alternatively, the algebraic structuralist of Section 4.2 may claim that the concrete representations of a geometric algebra $\mathscr{G}$ should not be read literally, but rather the structure defined by $\mathscr{G}$. Such a structuralist will claim that spacetime has the structure inherent in the abstract real Clifford algebra $\mathscr{C}_{(1,3)}$.

## 6. Spacetime as structure

The above review of alternative formalisms indicates that classical field-theoretic physics can be done without a 4-dimensional differentiable manifold, at least for most theories of interest. Minimally, this suggests that, if we desire to read classical field theories at their "face value", differentiable manifolds need not enter into our considerations: manifold substantivalism is not the only way to literally interpret a classical field theory. What does this suggest about the ontological status of spacetime? In particular, if we desire to be semantic realists with respect to classical field theories, what attitude should we adopt toward the nature of spacetime? One initial moral that can be drawn from the preceding discussion is that "fundamentalism" is in the eye of the beholder. In particular, all the alternative formalisms discussed above *disagree* on what the essential structure is that is minimally required to kinematically and dynamically support classical field theories.

### *6.1. Against fundamentalism*

Note first that the relations between the tensor formalism and the alternative formalisms reviewed above may be summarized as follows. Projective twistor space $\mathbb{PT}$ encodes the *conformal structure* ($M$, $\Omega\eta_{ab}$) of Minkowski spacetime (i.e., the metrical structure up to a multiplicative constant $\Omega$), with limited extensions to curved spacetimes. The dynamics of physical fields is encoded by geometrical structures on $\mathbb{PT}$ and its extensions. An Einstein algebra directly encodes the *differentiable structure* on $M$ (i.e., the points of $M$ imbued with differentiable and topological properties), and then encodes physical fields as derivations on this structure[40]. The Dirac algebra directly encodes the *metrical structure* $(M, \eta_{ab})$ of Minkowski spacetime, and then encodes physical fields and their dynamics as geometric functions on this structure (i.e., maps from $\mathscr{D}^1$ to subalgebras of $\mathscr{D}$).

A manifold substantivalist is a "point set fundamentalist". In the tensor formalism, this may seem a natural way to literally interpret spacetime: The point set of the manifold is the fundamental mathematical object, on which additional structures supervene. In particular, the moves to differentiable, conformal, and metrical structures are accomplished by adding more properties to the point set. On the other hand, proponents of alternative formalisms may claim that the manifold gives us *too much* as a representation of spacetime. In particular, they may charge one or more of the features of $M$ with the status of surplus mathematical structure in the context of classical field theories.

Proponents of twistors may claim that conformal structure is what is essential. They may claim that both the point set and the differentiable structure of $M$ are surplus: The point set can be reconstructed via the Klein Correspondence from twistors, while the differentiable structure is encoded in geometric/algebraic constructions over an appropriate twistor space. Moreover, twistor advocates will attempt to rewrite classical field theories in a conformally invariant way, hence they will also consider metrical structure as surplus.

Proponents of Einstein algebras may claim that differentiable structure is minimally sufficient to do classical field theory and view the point set of $M$ as surplus structure, and conformal and metrical structure as derivative.

Finally, proponents of geometric algebra may claim that metrical structure gives us everything we need for field theory, and view the point set, and the differentiable and conformal structures of $M$ as surplus. The point set is no longer needed to support fields, and the role played by differentiable structure is encoded directly in the Dirac algebra (in particular, in $\mathscr{D}^0$). There is also a precise sense in which conformal structure is derivative of $\mathscr{D}$: It turns out that twistors, as well as 2-component spinors, can be realized in the Dirac algebra. Lasenby et al. (1993) indicate how this is achieved by the following correspondences for the 2-spinor spaces $\mathbb{S}$, $\mathbb{S}'$ and twistor space $\mathbb{T}$:

$$\mathbb{S} = \{\forall\psi \subset \mathscr{D} : \psi = \kappa\tfrac{1}{2}(1+\sigma_3), \quad \text{for any } \kappa \in \mathscr{P}^+\}$$

$$\mathbb{S}' = \{\forall\psi \subset \mathscr{D} : \; \psi = -\omega \mathrm{i}\sigma_2\tfrac{1}{2}(1-\sigma_3), \quad \text{for any } \omega \in \mathscr{P}^+\}$$

$$\mathbb{T} = \{\forall Z \subset \mathscr{D} : \; Z = \phi + r\phi\gamma_0 \mathrm{i}\sigma_3\tfrac{1}{2}(1-\sigma_3), \quad \text{for any } \phi \in \mathscr{D}^+\}$$

[40] An original Geroch–Einstein algebra encodes local differentiable structure, whereas its commutative and non-commutative extensions may be said to encode global differentiable structure.

where $\mathscr{P}^+$, $\mathscr{D}^+$ are the even Pauli and Dirac subalgebras, and $r = \gamma_\mu x^\mu$[41]. The *GA* fundamentalist then may argue that, if spacetime is encoded by the Dirac algebra $\mathscr{D}$, then $\mathbb{S}$, $\mathbb{S}'$ and $\mathbb{T}$ are less fundamental than spacetime in the sense of being contained within $\mathscr{D}$. The main point, however, is that this cuts both ways: In the spinor formalism, Minkowski vector space (as encoded in $\mathscr{D}$) may be said to be derivative of $\mathbb{S}$ in the sense that it isomorphic to the real subspace $\text{Re}(\mathbb{S} \times \mathbb{S}')$; and, of course, in the twistor formalism, the points of (compactified) Minkowski spacetime $\mathbb{M}^c$ can be derived from geometric relations in $\mathbb{T}$.

The conclusion, then, is that what counts as fundamental and what counts as derivative, from a mathematical point of view, depends on the formalism.

## 6.2. *For structuralism*

The debate between these fundamentalisms revolves around what the essential structure of spacetime is that is necessary to support classical field theories: a point set, or differentiable, conformal, or metrical structures. But it does not revolve around how this structure manifests itself: in particular, what it is predicated on; or, in general, the nature of the basic mathematical objects that are used to describe it. This suggests adopting a structural realist approach to spacetime ontology.

Such spacetime structuralism, as motivated here, depends on prior semantic realist sympathies. It says: If we desire to be semantic realists with respect to classical field theories; i.e., if we desire to interpret such theories literally, or take them at their "face value", then we should be ontologically committed to the structure that is minimally required to kinematically and dynamically support mathematical representations of physical fields. Just what this structure is depends explicitly, for a semantic realist, on the formalism one adopts, as indicated above. Note, however, that this is not to say that essential structure is a matter of convention, in so far as the formalism one adopts generally is not a matter of pure convention. Rather, in the context of classical field theory, it will be influenced by inter-theoretical concerns; concerns, for instance, over which formulation of quantum field theory one adopts, or which approach to quantum gravity one adopts. Thus ultimately, the essential structure of classical field theory is empirical in nature, in so far as, ultimately, which extended theory (quantum field theory, quantum gravity) is correct is an empirical matter. What

[41] In the transcriptions for $\mathbb{S}$ and $\mathbb{S}'$, $\kappa$ and $\omega$ are the *GA* realizations of $SU(2)$ spinors and the factors $(1+\sigma_3)$ and $(1-\sigma_3)$ essentially realize chiral operators in $\mathscr{D}$ (the factor $i\sigma_2$ in $\mathbb{S}'$ realizes Hermitian conjugation). Thus elements of $\mathbb{S}$ and $\mathbb{S}'$ may be thought of as right- and left-handed spinors. (More precisely, they are right- and left-handed Weyl spinors in the Weyl representation of the Dirac operator algebra.) In the transcription for twistor space $\mathbb{T}$, a twistor in the *GA* formalism is realized as a position-dependent Dirac 4-spinor (in the Weyl representation). See Lasenby et al. (1993) for details.

the spacetime structuralist cautions against (in the here and now) is adopting an "individuals-based" ontology with respect to this structure. Conformal structure, for instance, can be realized on many different types of "individuals": manifold points, twistors, or multivectors, to name those considered in this essay. What is real, the spacetime structuralist will claim, is the structure itself, and not the manner in which alternative formalisms instantiate it.

As a form of realism with respect to spacetime, spacetime structuralism thus can be characterized by the following:

(a) It is not *substantivalism*: It is not a commitment to spacetime points.
(b) It is not *relationalism*: It does not adopt an anti-realist attitude toward spacetime[42].
(c) Rather, it claims spacetime is a real structure that is embodied in the world.

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[42] A traditional relationalist claims spacetime does not exist independently of physical objects (be they particles or fields). In this (perhaps limited) sense, relationalists are anti-realists with respect to spacetime. A spacetime structuralist claims that spacetime does exist independently of physical objects; but as a structure and not in the form of particular instantiations of that structure.

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