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Category-theoretic structure and radical ontic structural realism

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Abstract Radical ontic structural realism (ROSR) claims that structure exists independently of objects that may instantiate it. Critics of ROSR contend that this claim is conceptually incoherent, insofar as, (i) it entails there can be relations without *relata*, and (ii) there is a conceptual dependence between relations and *relata*. In this essay I suggest that (ii) is motivated by a set-theoretic formulation of structure, and that adopting a category-theoretic formulation may provide ROSR with more support. In particular, I consider how a category-theoretic formulation of structure can be developed that denies (ii), and can be made to do work in the context of formulating theories in physics.

Keywords Structural realism · Category theory · General relativity

1 Introduction

The aim of this essay is to defend radical ontic structural realism against the charge that it rests on an incoherent claim; namely, that there can be relations devoid of *relata* in the physical world. I will suggest that this claim is incoherent under a notion of physical structure informed by set theory, but not under a notion of physical structure informed by category theory. Section 3 argues for this in part by means of an analogy from general relativity. The debate over ontic structural realism concerns the ontological status of objects.[1] In general relativity, there is a similar debate over the

[1] Objects, in this sense, may or may not be considered individuals, depending on one's notion of individuation.

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ontological status of spacetime points. I'll argue that, for a significant family of solutions to the Einstein equations, one can speak meaningfully of spatiotemporal structure in the absence of spacetime points, provided one adopts an appropriate mathematical formalism. Section 4 discusses another example of structures in physics that cannot be articulated simply in terms of invariant properties predicated on objects. Finally, Sect. 5 identifies additional tasks that the category-theoretic radical ontic structural realist might address in order to further strengthen her position.

2 No relations without *relata*?

Radical ontic structural realism (ROSR, hereafter) is based on a denial of what French (2010, p. 178) calls "object oriented realism", which posits an ontology of objects and their properties and/or relations. ROSR, in contrast, claims that structure is what is real, and that structure consists of relations devoid of *relata* (French and Ladyman 2003). This claim has come under much criticism. Esfeld and Lam (2008, p. 31), for instance, acknowledge that one might posit the existence of abstract relations-as-universals without reference to *relata*, but "…when it comes to the physical world, the point at issue are concrete relations that are instantiated in the physical world and that hence are particulars in contrast to universals. For the relations to be instantiated, there has to be something that instantiates them…." With respect to the view that there are only relations without *relata*, Stachel (2006, p. 54) states: "As applied to a particular relation, this assertion seems incoherent. It only makes sense if it is interpreted as the metaphysical claim that ultimately there are only relations; that is, in any given relation, all of its *relata* can in turn be interpreted as *relations*." Wüthrich (2009, p. 1041) agrees with Stachel's assessment: "Taken at face value…[radical ontic structural realism] is clearly incoherent…". Finally, Dorato (2008, p. 21) states "I daresay that no ontic structural realist should be falling into the trap of accepting the view that 'relations can exist without *relata*'."

As Chakravartty (2003, p. 871) notes, criticism of this type assumes that there is a conceptual dependence between the notions of relation and *relata*, and to the extent that ROSR recommends a revision of such concepts, it cannot be faulted simply for denying this dependence. On the other hand, as Greaves (2009, pp. 17–18) suggests, the onus is still on ROSR to make good on just how such a dependence can be denied. Towards this goal, I will now consider a typical set-theoretic formulation of the notion of structure, and then compare it with a category-theoretic formulation. My claim will be that a conceptual dependence between relation and *relata* is suggested by the former, but not the latter.

2.1 Set theory versus category theory

If one adopts a set-theoretic formalism, then ROSR may indeed seem incoherent. Suppose, for example, that by "structure" we mean "isomorphism class of structured sets", $[\{X, R_i\}]$, where a structured set $\{X, R_i\}$ consists of a domain X of individuals together with a collection of n-ary relations R_i defined on it. The ontic structural realist's claim then is that the specification of the domain X of individuals is accidental to the concept of structure: what matters is the structure of the relations these arbitrary individuals enter into. Now suppose, to take the simplest example, by "binary relation R on X", we mean "subset of the Cartesian product $X \times X$". Insofar as the latter consists of all ordered pairs (x_1, x_2), where $x_1, x_2 \in X$, this definition makes ineliminable reference to the elements of X (let the ordered pair (x_1, x_2) be the set $\{x_1, \{x_1, x_2\}\}$). Hence if the *relata* of a relation associated with a structure are identified with the elements of its domain, then the set-theoretic definition of structure as an isomorphism class of structured sets makes ineliminable reference to *relata*. In general, one might argue that any set-theoretic definition of structure does likewise, insofar as membership "$\in$" is a primitive concept in set theory. This ineliminable reference to *relata* in set-theoretic definitions of structure subsequently suggests a conceptual dependence between structures and relations on the one hand, and *relata* on the other.

Now consider adopting a category-theoretic formalism to represent structure. In category theory, the primitives are objects, and morphisms between objects.[2] In particular, a category $\mathbf{C}$ consists of objects $A, B, \ldots$ and morphisms $f : A \to B, \ldots$. In addition, we require that for each object A, there be an identity morphism $1_A : A \to A$, which satisfies the Identity Laws $1_A \circ f = f$, and $f \circ 1_A = f$, for any morphism f with A as domain; and we require that there be composite morphisms $f \circ g : A \to C$ for each pair of morphisms of the form $f : A \to B, g : B \to C$, which satisfy the Associative Law $f \circ (g \circ h) = (f \circ g) \circ h$, for $h : C \to D$. It turns out that set theory can be formulated as a category, **Set**, in which the objects are sets and the morphisms are functions defined on sets. Moreover, for any given structured set, there is a category in which the objects are that type of structured set and the morphisms are functions that preserve the structure of the set (see Lawvere and Shanuel 1997 for elementary examples). This suggests that the intuitions of the ontic structural realist may be preserved by defining "structure" in this context to be "object in a category".

To what extent does such a category theoretic definition of structure eliminate reference to *relata*? As Bell (1988, p. 5) observes, "[i]n category theory many concepts formulated in terms of *elements* are instead formulated in terms of *arrows* [viz., morphisms]". In particular, the notion of an element of an object only makes sense in those categories with certain types of objects; namely, *terminal objects*. An object $\mathbf{1}$ in a category $\mathbf{C}$ is a terminal object of $\mathbf{C}$ if for each object X of $\mathbf{C}$, there is exactly one $\mathbf{C}$-morphism $X \to \mathbf{1}$. In categories with terminal objects, an *element* of an object A is then defined as a morphism $\mathbf{1} \to A$ from the terminal object to A. (In the category **Set**, the terminal object is the isomorphism class of singleton sets.) Thus, set-theoretic statements of the form "$x \in X$" ("x is an element of X") are translated into category-theoretic statements of the form "$x : \mathbf{1} \to X$" ("x is a morphism from the terminal object to X"). In the latter statement, there is no explicit reference to an "internal constituent" of the object X; rather, there are references to things external to X (i.e., the object $\mathbf{1}$ and the morphism x).

As another example, consider the concept of Cartesian product which underlies the set-theoretic concept of relation. In category theory, a product of an object X with

[2] In category theory, the term "object" has a specific mathematical use that is distinct from its use in the debate over ontic structural realism. Hopefully in the following the context will make it clear which use the term "object" is being put to.

itself is an object P together with a pair of morphisms $p_1 : P \to X$ and $p_2 : P \to X$, such that, for any object T with morphisms $f_1 : T \to X$, $f_2 : T \to X$, there is exactly one morphism $f : T \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$. One can demonstrate that in the category **Set**, such a product exists and is unique, and is given by the Cartesian product with its ineliminable reference to the elements of X (i.e., the morphisms $\mathbf{1} \to X$). In the general definition, however, there is no explicit reference to the elements of X. In general, one might say that category theory lacks the resources for direct reference to "internal" elements of an object. Thus in forming the definition of a product of an object with itself, we need to construct the right external "probe" (T, f_1, f_2, f) that directly encodes what in set theory is the "internal" pair structure of P. This suggests that the definition of structure as an object in a category does not make ineliminable reference to *relata* in the set-theoretic sense.

Two issues should be made clear at this point. The first concerns the extent to which set theory and category theory can be viewed as two *distinct* formalisms in which a notion of structure can be expressed. In the category theoretic formalism, one can represent a set by means of objects in the category **Set**. Similarly, in the set theoretic formalism, one can represent a category in terms of set-theoretic constructions. The point to be kept clear in this context is that the notion of a category can also be expressed in purely category-theoretic terms: setting aside foundational issues for the moment, a category can be specified without recourse to set-theoretic constructions. Thus, in particular, one should make a clear distinction between set theory on the one hand, and the category **Set** on the other. In general, one can distinguish between *external* and *internal* descriptions of a given category.[3] An external description is one framed in the language of set theory; whereas an internal description is one framed in the language "internal" to the category in question. Formally, internal descriptions gain traction in categories known as toposes (see, e.g., Döring and Isham 2011, p. 769; Heunen et al. 2009, p. 73). Whereas a structured set provides the mathematical structure for a formal semantics for classical logic, a topos provides the mathematical structure for a formal semantics for intuitionistic logic. Thus if we restrict ourselves to categories that are toposes, we have a formal means of distinguishing between set-theoretic discourse in which *relata* play an ineliminable role, and category-theoretic discourse in which *relata* are surplus.[4]

Given that set theory and category theory can be viewed in this way as distinct formalisms (or, if we restrict attention to toposes, distinct *languages*) in terms of which a concept of physical structure can be expressed, a second issue concerns how these formalisms compare with each other. In particular, one might ask, Is category theory

[3] Thanks to a referee for raising these issues.

[4] For the topos-theoretic contrast between internal and external descriptions, see, e.g., Mac Lane and Moerdijk (1992, p. 235), or Bell (1988, p. 105). Awodey (2010, p. 29) suggests a similar contrast but employs the terms "external" and "internal" in a different manner: "[Category-theoretic definitions] are characterizations of properties of objects and arrows [i.e., morphisms] in a category solely in terms of other objects and arrows, that is, just in the language of category theory. Such definitions may be said to be abstract, structural, operational, relational, or perhaps external (as opposed to internal). The idea is that objects and arrows are determined by the role they play in the category via their relations to other objects and arrows, that is, by their position in a structure and not by what they 'are' or 'are made of' in some absolute sense."

expressively equivalent to set theory in articulating an appropriate notion of physical structure? The suggestion above is that set theory has too much mathematical structure ("surplus" structure, if you will) in the form of *relata* when it comes to representing structures in the physical world, and that category theory is to be preferred since it removes this unnecessary surplus.

2.2 The elimination of relata in name only?

One might object to the above suggestion in the following way: Category theory eliminates reference to *relata* only in name. Instead of calling the *relata* associated with a structure "elements of the structure's domain", as in set theory, category theory calls them "morphisms from the terminal object". Assumedly, or so the objection goes, any given set theoretic structure will have a category theoretic analog, and however many *relata* the former is associated with, so the latter will be associated with the same number of morphisms from the terminal object. The argument against the radical ontic structural realist then gets translated from the slogan *no relations without relata* to the slogan *no objects* (*of the relevant sort*) *without morphisms from the terminal object*.

My response to this objection is to agree that set-theoretic *relata* do have correlates in category theory, but to point out that, in many cases, these correlates are not essential to the articulation of the relevant structure. In particular, category-theoretic objects need not be structured sets, and the structure encoded in objects that *do not* have structured set correlates does not depend in an essential way on their elements. Now in order to guarantee that this mathematical fact has physical significance, it behooves a category theoretic ROSRer to provide examples of "non-structured set" objects that have roles to play in articulating relevant notions of structure in physics. The next two sections attempt to make headway on this task.

3 An analogy from general relativity

My defense of ROSR rests on the suggestion that moving from a set-theoretic formalism to a category-theoretic formalism supports an ontology of structure in which the articulating role that *relata* play in the former is eliminated. Similarly, in general relativity (GR, hereafter), I will now argue, moving from the tensor formalism to the Einstein algebra formalism supports an ontology of spatiotemporal structure in which the articulating role that spacetime points play in the former is eliminated.

Typically, spacetime points are represented in the tensor formalism by the points of a differentiable manifold M. Models of GR in this formalism consist of a pair (M, g_{ab}), where g_{ab} is a metric field defined on M and satisfying the Einstein equations. From a set-theoretic point of view, such a tensor model is a structured set. It consists of a domain of *relata* (manifold points) on which are predicated topological, differentiable, and metrical properties.

General relativity can also be formulated in terms of Einstein algebras. An Einstein algebra (alternatively, a model of GR in the Einstein algebra formalism) consists of a pair (C, g) where C is a commutative ring satisfying three axioms, and g is a multilinear map defined on the space of derivations of C and its dual space, that satisfies the algebraic correlate of the Einstein equations.[5] A 1-1 correspondence between Einstein algebra (EA) models and tensor models exists, based on the 1-1 correspondence between the points of a differentiable manifold and the maximal ideals of the commutative ring of smooth functions defined on M.[6] This correspondence allows all the relevant tensorial objects defined on M in tensor models of GR to be translated into appropriate algebraic objects defined in an Einstein algebra model. Thus the Einstein algebra formalism is as expressive as the tensor formalism in the sense that any tensor model of GR corresponds to an EA model of GR.

Now, arguably, tensor and EA models disagree at the level of "*relata*-based" ontology. The *relata* associated with tensor models are manifold points, insofar as manifold points are the objects of predication in tensor models (they are the *relata* on which spatiotemporal properties are predicated). The *relata* associated with Einstein algebra models are, under the correspondence mentioned above, maximal ideals of smooth functions, which, at least at face-value, are mathematical entities distinct from manifold points. However, the isomorphism between these models suggests they agree at the level of structure. In general, tensor models of GR are invariant under the actions of $Diff(M)$, the group of diffeomorphisms on M (but see Sect. 3.1 below). EA models of GR share this invariance property, although in the EA formalism, it gets translated into actions of the group of homomorphisms that leave invariant (C, g). In both cases, the structure associated with these transformations may be identified as *differentiable structure*. In tensor models, this is predicated on the points of M, whereas in EA models, it is associated with the structure of a commutative ring of smooth functions on M.

Insofar as the *relata* associated with tensor models are distinct from those associated with EA models, in adopting the EA formalism, we eliminate explicit reference to manifold points. On the other hand, one might question whether this is an elimination of manifold points in name only. Given the 1-1 correspondence between tensor models and EA models, to every manifold point in the former, there corresponds a maximal ideal in the latter (and vice-versa). Thus any reference to a manifold point in a tensor model of GR will be translatable in a 1-1 fashion into a reference to a maximal ideal in an EA model. A set-theoretic ROSRer may initially be happy with this, insofar as it is a demonstration that tensor models and EA models belong to the same isomorphism class of structured sets, and hence encode the same structure. But to the extent that tensor models depend ineliminably on manifold points, so EA models depend ineliminably on maximal ideals. In what sense, then, can we say that the differentiable structure associated with these models stands on its own, independently of *relata*? I'll now argue that, at least for some solutions to the Einstein equations, the move to the Einstein algebra formalism is in fact a move that non-trivially eliminates manifold points (and their EA correlates), but retains differentiable structure.

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5 For details, see Heller and Sasin (1995, p. 3644). The original formulation was given in Geroch (1972).

6 A maximal ideal of a commutative ring is the largest subset of the ring closed under the ring product. Each point of a differentiable manifold M corresponds to a maximal ideal of smooth functions on M that vanish at that point.

3.1 Asymptotic boundary conditions

The class of solutions I have in mind is characterized by asymptotic boundary conditions. Examples include solutions to the Einstein equations that behave asymptotically like spacetimes of constant curvature, and solutions involving certain types of curvature singularities.[7] In the tensor formalism, such boundary conditions can be represented by encoding them in absolute objects defined on a boundary space ∂M and attaching it to the manifold M. The result is a manifold with boundary $M' = M \cup \partial M$. It is then the case that, while tensor models (M, g_{ab}) without such boundary conditions are invariant under the group $Diff(M)$ of diffeomorphisms on M, tensor models (M', g_{ab}) *with* such boundary conditions are in general invariant under the subgroup $Diff_c(M)$ of diffeomorphisms on M with compact support. A diffeomorphism is in $Diff_c(M)$ just when there is a compact region of M outside of which it is the identity (Belot preprint, pp. 7, 13–14). One can think of such transformations as "local" diffeomorphisms ("local" in the sense of being, possibly, non-trivial only in a localized compact region of M). Intuitively, such local diffeomorphisms are guaranteed to preserve the local structure of M and trivially preserve the boundary space ∂M (since they are the identity there). Elements of the larger group $Diff(M)$, on the other hand, are only guaranteed to preserve the structure of M and may fail to preserve the absolute structure of the boundary space. Thus, in general, there are no non-trivial morphisms (i.e., transformations) that preserve *both* M and M' (no diffeomorphism on M is guaranteed to extend non-trivially to a diffeomorphism on M'). Technically this means that manifolds and manifolds with boundaries belong to different categories.

On the other hand, asymptotic boundary conditions of this type can be imposed on Einstein algebra models of GR in two steps (Heller and Sasin 1995, p. 3657). One first replaces the *ring* $C \cong C^\infty(M)$ of real-valued smooth functions on M with a *sheaf* $\mathcal{C} \cong C^\infty(M')$ of real-valued smooth functions on the corresponding M'. One then replaces the *Einstein algebra* (C, g) defined on M with a *sheaf of Einstein algebras* $(\mathcal{C}, g)$ defined on M'. A sheaf of Einstein algebras can be thought of as a collection of Einstein algebras indexed by the open regions on M induced by the topology on M' (see Sect. 3.2 below for more detail). It turns out that, as algebraic objects, (C, g) and $(\mathcal{C}, g)$ belong to the same category, what Heller and Sasin (1995, p. 3647) have dubbed the category of *Einstein structured spaces*. In particular, one can define morphisms that preserve the structure of *both* (C, g) and $(\mathcal{C}, g)$.[8]

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[7] Examples of the former include solutions that are asymptotically spatially flat, and solutions that behave asymptotically like de Sitter, anti-de Sitter, or Minkowski spacetime (see, e.g., Belot preprint, pp. 49–55 for details and references). Examples of the latter include spacetimes with b-incomplete curves (see, e.g., Earman 1995, p. 36).

[8] A *structured space* is a pair $(M, \mathcal{C})$, where M is a topological space and $\mathcal{C}$ is a sheaf of real continuous functions on M satisfying the following condition (*closure with respect to composition with smooth Euclidean functions*): For any open set U in the topology τ on M and any functions $f_1, \ldots, f_n$ in $\mathcal{C}(U)$, and any smooth function ω on $\mathbb{R}^n$, the composite $\omega \circ (f_1, \ldots, f_n)$ is in $\mathcal{C}(U)$ (Heller and Sasin 1995, p. 3645). Now let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be structured spaces. A continuous mapping $f : M \to N$ is said to be *smooth* if, for any cross section g in $\mathcal{D}(U)$, the composite $g \circ (f | f^{-1}(U))$ is in $\mathcal{C}(f^{-1}(U))$ (Heller and Sasin 1995, p. 3647). One can now show that the collection of structured spaces, identified as objects, and smooth mappings, identified as morphisms, forms a category. An *Einstein structured space* is a structured space on which is defined the correlate of the Einstein equations.

3.2 Sheaves and relata

In the debate over the ontological status of spacetime points, the concern with employing an Einstein algebra to describe a solution of the Einstein equations was that it eliminates reference to manifold points in name only. Does a sheaf of Einstein algebras fair any better? Consider first a more simple type of sheaf; namely, a sheaf of sets (see, e.g., Jozsa 1984, pp. 68–69).[9] Technically, a *sheaf $\mathcal{S}$ of sets over a topological space X* is an assignment of a set $\mathcal{S}(U)$ to each nonempty open set $U \subseteq X$, and a restriction map $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ whenever $U \supseteq V$, that together satisfy the following conditions:

(i) If $U \subseteq V \subseteq W$, then $\rho_{WV} = \rho_{VU} \circ \rho_{WV}$.
(ii) Let $\{U_i\}$ be any open cover of U and let $U_{ij} = U_i \cap U_j$. If $\sigma_i \in \mathcal{S}(U_i)$ is a collection of elements such that $\rho_{U_i U_{ij}}(\sigma_i) = \rho_{U_j U_{ij}}(\sigma_j)$ for all i, j, then there is a unique element $\sigma \in \mathcal{S}(U)$ such that $\sigma_i = \rho_{UU_i}(\sigma)$.

The elements $\sigma \in \mathcal{S}(U)$ are referred to as *sections* of $\mathcal{S}$ over U. Condition (ii) is the requirement that any collection of sections of $\mathcal{S}$ over an open cover of U can be patched together to form a unique section over U, as long as they agree when restricted to intersections. The identification of elements with sections allows one to define an element of the sheaf itself. Technically, $\mathcal{S}$ is not a set; rather, it is a collection of sets indexed by the open regions of X. A *global section* of $\mathcal{S}$ is an assignment of an element of $\mathcal{S}(U)$ to each open region U. Such global sections may be identified as the elements of $\mathcal{S}$. Category theory makes this more precise: Let **Sh**(***X***) be the category of sheaves of sets over X.[10] Then the global sections of a given object $\mathcal{S}(X)$ in this category correspond to its elements in the category-theoretic sense (see Sect. 2.1 above). It then turns out that a sheaf of sets in general need not possess global sections, and even when it does, these fail to uniquely characterize it.

This situation should be compared with the case of the category **Set** of sets. **Set** can be obtained from **Sh**(***X***) by identifying X as the one-element topological space. The collection of sets characterized by any object $\mathcal{S}$ in **Sh**(***X***) then degenerates to a single set, and the global sections of $\mathcal{S}$ are simply the elements (in the set-theoretic sense) of the corresponding set. These elements completely characterize the set.

A slight generalization of a sheaf of sets over X replaces the sets with structured sets. One obtains a sheaf of structured sets over X, and one can then form the category of sheaves of structured sets over X, with a corresponding category of the appropriate type of structured set as the degenerate case in which X is the one-element topological space. The global cross sections of the latter are "structured" elements; i.e., elements of a set imbued with the invariant properties that define the type of structured set. Again, the existence of such structured elements for the degenerate set-theoretic case does not guarantee the existence of global cross sections for the general sheaf-theoretic case.

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[9] The following is an "external" description of a sheaf of sets over a topological space (i.e., a description in set-theoretic terms). For an "internal" description, see, e.g., Mac Lane and Moerdijk (1992, p. 66).

[10] The morphisms of this category are set maps that preserve the structure of the restriction maps ρ (Jozsa 1984, p. 69).

In the Einstein algebra formalism, a sheaf of Einstein algebras is an assignment of an Einstein algebra (a particular type of structured set) to every open region of a differentiable manifold M.[11] The degenerate case is given by a single Einstein algebra (take M to be a one-element manifold). A section of a sheaf of Einstein algebras over an open region U of M is an element of the Einstein algebra assigned to U; namely, it is a maximal ideal of that algebra. And again, the fact that each Einstein algebra admits elements of this nature does not guarantee that a sheaf of Einstein algebras does, insofar as the sheaf may not admit global sections.

3.3 Spatiotemporal structure sans relata

The upshot of this example seems to be the following. In both tensor models and EA models of GR, the invariant structure being represented is differentiable structure. In the case of tensor models, one might further characterize this structure as *local* differentiable structure (i.e., that which is encoded in $Diff_c(M)$), in order to cover cases of solutions both with and without asymptotic boundary conditions. This local differentiable structure is predicated on the points (*relata*) of a differentiable manifold M. In contrast, the structure associated with Einstein algebra models might be referred to as *global* differentiable structure. This is the structure associated, in general, with a sheaf of Einstein algebras, which cannot be said to be predicated on the point-correlates (i.e., maximal ideals) of any given Einstein algebra. Rather, there's a sense in which it is a global feature of the sheaf. In category-theoretic terms, this structure is encoded in the objects of the category of Einstein structured spaces.

The distinction between tensor models and EA models can be summarized by the observation that the former can be characterized as structured sets, while the latter, in general, cannot. I'd like to draw two conclusions from this distinction:

1. First, the point correlates (i.e., maximal ideals) in Einstein algebra models of GR do not play an essential role in articulating the relevant notion of structure (i.e., global differentiable structure).
2. And second, Einstein algebra models of GR provide a more unifying description of phenomena in GR, insofar as they belong to a single category. Tensor models of GR (at least for the examples of solutions with and without asymptotic boundary conditions) belong to two distinct categories.

This suggests the following analogy with radical ontic structural realism:

1′. First, the correlates of set-theoretic *relata* in category theory do not play an essential role in articulating the relevant notion of structure, insofar as, in general, objects in a category need not be structured sets.

2′. And second, this notion of structure does actual work in providing a more unifying description of physical phenomena.

[11] Technically, the open regions are identified as those that form a topology induced by the topology of a manifold with boundary (Heller and Sasin 1995, p. 3657). In general, a sheaf over a base space is an assignment of an algebraic object (set, algebra, etc.) to the open regions of the base space together with a restriction map that satisfies appropriate generalizations of conditions (i) and (ii) above.

The use of sheaves of Einstein algebras in general relativity provides one example of Claim (2′). A category-theoretic radical ontic structural realist may view the global differentiable structure associated with the objects in the category of Einstein structured spaces as representing free-standing spatiotemporal structure that exists independently of (the algebraic correlates of) spacetime points.

4 How to do category-theoretic physics

I'd now like to briefly look at two more examples of categories that have potential use in physics and whose objects do not correspond to structured sets (these examples are discussed in Baez 2006, pp. 246–247). The first is the category ***n*Cob** with $(n-1)$-dimensional compact oriented manifolds as objects and n-dimensional oriented cobordisms between such manifolds as morphisms.[12] The second example is the category **Hilb** with finite-dimensional Hilbert spaces as objects and bounded linear operators as morphisms. These differ from the category **Set** of sets (with functions as morphisms) in the following three respects.

(i) First, the objects of ***n*Cob** and **Hilb** cannot be considered structured sets, insofar as their morphisms are not simply functions that preserve the relevant set-theoretic notion of structure associated with them. Set-theoretically, the functions that preserve the structure of an $(n-1)$-dim topological manifold are homeomorphisms (i.e., maps that preserve the topological properties of points). But the morphisms in ***n*Cob** are not even functions. Set-theoretically, the functions that preserve the structure of a Hilbert space are unitary operators that preserve the inner-product. The morphisms in **Hilb** in contrast are general bounded linear operators that do not necessarily have to be unitary. (Baez 2006, p. 251, defines an inner-product on the objects in **Hilb** in terms of an adjoint operation, thus turning **Hilb** into a *-category: see (iii) below.)

(ii) Second, unlike **Set**, the categories ***n*Cob** and **Hilb** are *monoidal* categories. This means they admit a tensor product but not a Cartesian product. In particular, in both of these categories, for any pair of objects H, K, there is an object $H \otimes K$ called the tensor product of H and K, but there are no morphisms $p_1 : H \otimes K \to H$ and $p_2 : H \otimes K \to K$ with the properties of a Cartesian product (Baez 2006, p. 257).

(iii) Third, unlike **Set**, the categories ***n*Cob** and **Hilb** are *-categories. This means they admit a morphism * that sends each morphism $f : X \to Y$ to a morphism $f^* : Y \to X$ called the "adjoint" of f and satisfying $1_X^* = 1_X$, $(f \circ g)^* = g^* \circ f^*$, and $f^{**} = f$ (Baez 2006, p. 251).

Both ***n*Cob** and **Hilb** admit terminal objects, and hence a well-defined notion of an element of an object. As one might expect, the elements of ***n*Cob** objects are manifold points, and the elements of **Hilb** objects are vectors. But, insofar as the objects of

12 A cobordism in this context can be identified with an oriented manifold with boundary. The intended interpretation of ***n*Cob** identifies its objects as representing $(n-1\text{dim})$ spaces at different instances in time, and its morphisms as representing (n-dim) segments of spacetime connecting these spaces.

these categories are not structured sets, their elements are not essential in articulating the relevant notions of structure. Again, because the objects of these categories are not structured sets, the "properties" of their elements are not what get preserved under the morphisms. Thus the structure associated with the objects in ***n*Cob** and **Hilb** is arguably more general than that associated with their set-theoretic counterparts. In other words, the category-theoretic definitions of $(n-1)$-dim topological manifold and Hilbert space, as provided by the categories ***n*Cob** and **Hilb**, are more general than the set-theoretic definitions. Baez (2006) further argues that this generality is more than cosmetic: Baez sees the similarities between ***n*Cob** and **Hilb**—in particular, those features mentioned above that distinguish them from **Set**—as suggestive of how GR and quantum theory might be reconciled. Briefly, ***n*Cob** and **Hilb** play essential roles in a category-theoretic formulation of topological quantum field theories, which have been viewed by some authors as a method of reconciling the background independent nature of GR with quantum field theory.[13] One might view this as one way that the generality associated with the notions of structure in ***n*Cob** and **Hilb** has the potential to do actual work in articulating a notion of structure that addresses a key issue in physics.

5 What the category-theoretic radical ontic structural realist must do

Of course if the types of structures that ROSR is (or should be) concerned with are all of the structured set type (and hence depend ineliminably on reference to *relata*), then adopting a category-theoretic definition of structure would not be all that helpful. I've suggested that there are, in fact, non-trivial examples of structures in physics that are not of the structured set variety. On the other hand, one might also argue that the generality afforded by the category-theoretic definition of structure is a moot point if it turns out that category theory presupposes set theoretic concepts. If this is the case, then categories are really just sets in disguise, even those categories that do not have structured sets as objects; thus there would be no greater expressiveness to be associated with category theory. In particular, the claim would be that the membership relation really is a primitive in category theory, the examples in this essay none withstanding; hence a category-theoretic definition of structure would not, ultimately, break free of *relata*. Thus there is still work to be done for the category-theoretic radical ontic structural realist:

(1) She should provide a rationale for the fundamentality of category theory over set theory. For instance, Kraus (2005, p. 114) claims the following:

> The reason [ontic structural realists] don't use category theory is still not clear to me, but perhaps this is due to the fact that from an intuitive point of view a category is nothing more than an ordered pair (hence a set) whose elements are a collection of objects (the structures) and a collection whose elements are called

[13] More precisely, a topological quantum field theory can be defined as a functor from ***n*Cob** to **Hilb**; i.e., a map that assigns to each object in ***n*Cob**, an object in **Hilb**, and to each morphism in ***n*Cob**, a morphism in **Hilb**, in such a way that composition of morphisms is preserved, as is the identity morphism (Baez 2006, p. 248).

morphisms (both concepts of course are subjected to adequate postulates). That is, even in category theory we are not completely free from the intuitive notion of sets.

If category theory can be shown to be more fundamental than set theory, this argument is blunted. The fact that a category can be presented as an ordered pair would reduce to the fact that a category can be presented as a category. More generally, the issue of fundamentalism concerns not just the practice of using set theory as a meta-language for category theory, but the status of the referents of this meta-language: are these referents ultimately sets or categories? Awodey (2010, p. 24) for instance observes that while the axioms of set theory are typically taken as making existential claims about a single universe of sets, the axioms of category theory are typically taken as definitions of things which are assumed to exist in some foundational system. If one assumes the foundational system is that of set theory, then issues arise in category theory concerning the size of categories: one may have to distinguish between "small" categories in which the objects and morphisms form sets, and "large" categories in which they do not. On the other hand, these concerns are less pressing if the foundational system is taken to be category theory itself. It is this concern over the fundamentality of category theory that the category-theoretic ROSRer must address. This on-going debate in the philosophy of mathematics deserves more space than can be provided here (see, e.g., Pedroso (2009) who addresses standard charges in the literature against category-theoretic fundamentalism).
(2) Second, the category-theoretic ROSRer should also provide additional category-theoretic reformulations of theories in physics that explicitly do not depend on structured sets. Döring and Isham (2011) are engaged in this project in the context of theories in quantum physics (see also Isham and Butterfield 2000; Heunen et al. 2009), and Baez (2006) has argued against set-theoretic intuitions in formulating approaches to quantum gravity.
(3) Finally, a deeper concern might be identified with the examples given by these authors (as well as those discussed in this essay). The fundamentality of category theory would be a moot point if it turned out that the majority of structures in the physical world *of physical relevance* are better represented by set-theoretic constructions. Critics of ROSR might argue that the abstract spatiotemporal structure associated with solutions to the Einstein equations with asymptotic boundary conditions is not relevant when it comes to making concrete observations and/or measurements of spatiotemporal phenomena, and perhaps doubly so for topological quantum field theories. Again, as Esfeld and Lam (2008, p. 31) stress, "…the point at issue are concrete relations that are instantiated in the physical world and that hence are particulars in contrast to universals".

There are two ways one might voice this worry. The first is to view it as a concern over whether an adequate *theory of measurement* requires set-theoretic constructions; in particular, explicit and ineliminable reference to *relata*. Theories of measurement, assumedly, are where the abstract rubber of theories in physics hits the concrete road of empirical tests, and the latter, so the objection would go, require representational schemes that make ineliminable reference to *relata*. While this concern deserves an extended treatment beyond the scope of this essay, I can think of two immediate responses. First, whether or not an adequate theory of measurement requires ineliminable reference to *relata* should be an empirical question. As such, it should not be pre-judged by intuitions informed (implicitly, perhaps) by a particular representational scheme (viz., set theory). Second, even if it turns out empirically that an adequate theory of measurement requires ineliminable reference to *relata*, it does not necessarily follow that a theory of the physical structures being measured does likewise. Conceivably, physical structures could be globally free-standing (in the sense of being independent of localized *relata*) in general, but instantiate locally during interactions involving measurement processes. Classically, the latter can be represented set-theoretically in terms of the instantiation of properties in *relata*; but who (other than an instrumentalist, perhaps) would expect that measurement interactions uniquely determine the ontological commitments of the physical systems that undergo them? Many philosophers of physics, for instance, believe quantum field theories should be interpreted as referring to globally-extended quantum fields, as opposed to localizable particles (or "quanta", if you will). Under this received view, a quantum field is capable of instantiating itself in the form of localized particles/*relata* in particular circumstances (scattering experiments, for instance); but such instantiations should not be taken to uniquely determine the ontological status of the field itself. This is not to say that quantum fields are in some sense inherently structural (nor is it to agree with the received view); rather, it is to suggest that, while ontological commitments may be informed by theories of measurement, they should not be completely determined by them.

A second way to voice the worry associated with Esfeld and Lam is to view it as a concern over whether, from a purely metaphysical point of view, it is coherent to claim that physical structures can exist independently of objects that instantiate them. The worry here is that it is one thing to demonstrate that, from a formal point of view, category theoretic *representations* of structure are independent of *relata*, and another thing to argue that structures in the physical world are likewise independent of *relata*.[14] Towards assuaging this concern, consider first the standard argument in support of ROSR. This argument is based on an appeal to *metaphysical underdetermination* in the context of quantum mechanics: ROSRers claim that, if we take the ontology of quantum mechanics to be one of objects and properties, then whether these objects are individuals or non-individuals is underdetermined by the theory; whereas if we take the ontology to consist of relations devoid of *relata*, no such underdetermination occurs (see, e.g., Ainsworth 2010, p. 52, for this reconstruction). As Pooley (2006, p. 97) notes, metaphysical underdetermination occurs when a *single formulation* of a theory admits multiple interpretations that involve incompatible ontologies (in this case an ontology of individuals versus an ontology of non-individuals). The example of general relativity in Sect. 3 above suggests a different type of underdetermination, one that Pooley refers to as "Jones underdetermination".[15] This occurs when a single theory admits *different formulations* that, under a realist interpretation, suggest incompatible ontologies. Section 3 suggested that, whereas the tensor formulation of

[14] Thanks to a referee for stressing the importance of this distinction.

[15] Pooley (2006, p. 97) calls this "Jones Underdetermination" after its description in Jones (1991).

general relativity supports an ontology of objects and properties/relations, the Einstein algebra formulation of general relativity supports an ontology of structure devoid of objects. The current essay thus motivates ROSR by examples of different formulations of a single theory that disagree not just over the ontological status of individuals, but over the ontological status of objects, *regardless* of whether or not they are individuals.[16] Thus, regardless of whether spacetime points are considered to be individuals, Sect. 3 suggested that they are inessential in representations of spatiotemporal structure as described by general relativity in the Einstein algebra formalism.

The inference to ROSR from Jones underdetermination depends on some form of semantic realism with respect to theories in physics (i.e., it assumes that we should take the claims made by theories in physics at their face value). And it assumes a naturalistic approach to metaphysics; one in which metaphysical commitments are informed by contemporary theories in physics, say, as opposed to pre-theoretic intuitions (informed, perhaps, by out-dated physics). In particular, it suggests that the manner in which some contemporary theories in physics represent natural phenomena is in terms of structure devoid of *relata*; and it suggests that we should take these representations at their face value and accommodate them into what we take to be the ontology of the world; as opposed to attempting to reconcile them with pre-theoretic intuitions (informed, perhaps, by set-theoretic representations of natural phenomena) that implicitly privilege object oriented ontologies.

6 Conclusion

This essay has argued that a definition of structure as an object in a category does not, in general, depend essentially on (set-theoretic) *relata*, insofar as category-theoretic objects need not, in general, be structured sets. In particular, the structure associated with such objects cannot be articulated simply in terms of invariant properties instantiated by arbitrary elements of a domain of individuals. Moreover, such (non-structured set) objects have roles to play in representing relevant notions of structure in contemporary physics. This suggests that radical ontic structural realism based on such a category-theoretic definition of structure avoids the charge that it rests on an incoherent claim; namely, that there can be relations devoid of *relata*.

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[16] ROSR thus rejects versions of ontic structural realism that retain a “thin” notion of object; i.e., one in which individuality is conferred in structural terms (see, e.g., French 2010, p. 182).

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