Recursive Undecidability and Incompleteness of $\mathcal{N}$

**Prop:** If $\mathcal{N}$ is consistent and recursively decidable, then any recursively enumerable subset of $\mathbb{N}$ is recursive.

**Proof:** Show that if $\mathcal{N}$ is consistent and recursively decidable, it can be used to construct a method that decides membership in any recursively enumerable subset of $\mathbb{N}$.

Suppose $A$ is a recursively enumerable subset of $\mathbb{N}$.

**Then:** There is a recursive function $f$ such that $A = \{ n \in \mathbb{N} : n = f(m), m \in \mathbb{N} \}$.

**Thus:** Since $f$ is recursive and therefore expressible in $\mathcal{N}$, there is a $\text{wf } A(x_1, x_2)$ such that

(i) If $f(m) = n$, then $\vdash_{\mathcal{N}} A(0^{(m)}, 0^{(n)})$ or $\vdash_{\mathcal{N}} A(0^{(m)}, 0^{(n)})$, then $f(m) = n$

(ii) If $f(m) \neq n$, then $\vdash_{\mathcal{N}} \sim A(0^{(m)}, 0^{(n)})$ or $\vdash_{\mathcal{N}} \sim A(0^{(m)}, 0^{(n)})$, then $f(m) = n$

**Now:** If $\mathcal{N}$ is recursively decidable, then for any $\text{wf } A$, there is an effective method that determines if $A$ is or is not a theorem (by Church's Thesis).

**So:** (i) and (ii) determine, for any $n \in \mathbb{N}$, if $n$ is or is not in the set $A$. For any $n$, $n$ is not in $A$ just when a certain $\text{wf}$ is not a theorem of $\mathcal{N}$. And $n$ is in $A$ just when the negation of this $\text{wf}$ is not a theorem.

**Note:** If $\mathcal{N}$ is consistent, then this method will always work: there will be no $n$ that both is and is not in $A$, since for any $\text{wf } A$, we cannot have both $\vdash_{\mathcal{N}} A$ and $\vdash_{\mathcal{N}} \sim A$.

**Corollary:** If $\mathcal{N}$ is consistent, then it cannot be recursively decidable.

**Proof:** Suppose $\mathcal{N}$ is consistent and recursively decidable.

**Then:** Any recursively enumerable set is recursive (above Prop).

**But:** $K$ is a recursively enumerable set that is not recursive. (Prop. 7.30.)

**Alternative Proof of Recursive Undecidability of $\mathcal{N}$**

Suppose $\mathcal{N}$ is consistent and recursively decidable.

**Now:** Enumerate all $\text{wfs}$ of $\mathcal{L}_\mathcal{N}$ with one free variable: $A_0(x), A_1(x), ...$

**Next:** Define a 1-place relation $D$ on $\mathbb{N}$ by:

$$D(n) \text{ holds } \iff \neg \vdash_{\mathcal{N}} A_n(0^{(m)})$$

**Then:** Since $\mathcal{N}$ is assumed to be recursively decidable, there is an effective method that determines if $D(n)$ holds; namely, $D(n)$ holds if and only if the $\text{wf } A_n(0^{(m)})$ is a theorem of $\mathcal{N}$.

**So:** By Church's Thesis, $D$ is recursive.

**Thus:** $D$ is expressible in $\mathcal{N}$, say by the $\text{wf } A^D(x)$ such that

(i) If $D(n)$ holds, then $\vdash_{\mathcal{N}} A^D(0^{(m)})$

(ii) If $D(n)$ doesn't hold, then $\vdash_{\mathcal{N}} \sim A^D(0^{(m)})$

**Now:** $A^D(x)$ must appear in the list of $\text{wfs}$ with one free variable, say $A^D(x) = A_m(x), n \in \mathbb{N}$.

**So:** For the case $n = m$, we have:

1. If $D(m)$ holds, then $\vdash_{\mathcal{N}} \sim A_m(0^{(m)})$ (definition of $D$)
2. If $\vdash_{\mathcal{N}} \sim A_m(0^{(m)})$, then $D(m)$ holds (definition of $D$)
3. If $D(m)$ holds, then $\vdash_{\mathcal{N}} A_m(0^{(m)})$ (expressibility of $D$ (i))
4. If $D(m)$ doesn't hold, then $\vdash_{\mathcal{N}} \sim A_m(0^{(m)})$ (expressibility of $D$ (ii))

**Now:** If $\mathcal{N}$ is consistent, then (1) and (3) entail $D(m)$ cannot hold.

**But:** If $D(m)$ doesn't hold, then (2) entails $\not\vdash_{\mathcal{N}} \sim A_m(0^{(m)})$, whereas (4) entails $\not\vdash_{\mathcal{N}} \sim A_m(0^{(m)})$.

**So:** If $\mathcal{N}$ is consistent, it cannot be recursively decidable.
**Prop.** If a first order system $S$ is complete, then it is recursively decidable.

**Proof:** Suppose $S$ is complete.

**Then:** If $S$ is inconsistent, it is recursively decidable (the set of $S$-theorems will be identical to the set of all $wfs$, which is recursively decidable).

**So:** Suppose $S$ is consistent.

**Then:** The following is an effective method to determine if a $wf$ $A$ is a theorem of $S$:

1. Enumerate the theorems of $S$.
2. Search list until either $A$ or $\sim A$ is found.
3. If $A$ is found, it is a theorem of $S$. If $\sim A$ is found, $A$ is not a theorem of $S$ (by completeness).

**Thus:** By Church's Thesis, the characteristic function for the set of $G$-numbers of $S$-theorems is recursive; hence the set of $G$-numbers of $S$-theorems is recursive; hence $S$ is recursively decidable.

**Comment:** Recall that the set of theorems of any (recursively axiomatizable) first order system $S$ is recursively enumerable. And this entails that, for any $wf$ $A$, if $A$ is an $S$-theorem, then it will occur somewhere in the list of theorems. But if $A$ is *not* an $S$-theorem, no effective search of the list will halt. If $S$ is complete, then one can search for either $A$ or $\sim A$; and such a search is guaranteed to halt eventually.