Intertheoretic Implications of Non-Relativistic Quantum Field Theories

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1. NQFTs and Particles
2. Newtonian Quantum Gravity
3. Intertheoretic Relations
1. NQFTs and Particles

- **Relativistic quantum field theory (RQFT) =** A QFT invariant under the symmetries of a Lorentzian spacetime.

- **Non-relativistic quantum field theory (NQFT) =** A QFT invariant under the symmetries of a classical spacetime.
1. NQFTs and Particles

**Arena for RQFTs:** Lorentzian spacetime \((M, g_{ab})\).

- \(g_{ab}\) - pseudo-Riemannian metric with Lorentzian signature \((1, 3)\).
- \(\nabla_a g_{bc} = 0\) for unique \(\nabla_a\) (compatibility)

**Ex. 1:** Minkowski spacetime (spatiotemporally flat): \(R^a_{\ bcd} = 0\).

- No unique way to separate time from space:

  - Symmetry group generated by \(\xi_x g_{ab} = 0\). (Poincaré group)
1. NQFTs and Particles

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**Ex. 1:** Minkowski spacetime (spatiotemporally flat): \(R^{a}_{bcd} = 0\).

**Ex. 2:** Vacuum Einstein spacetime (Ricci flat): \(R_{ab} = 0\).

**Comparison:**

- *Different* metrical structure, *different* curvature, same metric signature \((i.e., "in the small", isomorphic to Minkowski spacetime).
- Different types of RQFTs, in flat (Minkowski) and curved Lorentzian spacetimes.
1. NQFTs and Particles

**Arena for NQFTs:** Classical spacetime \((M, h^{ab}, t_{ab}, \nabla_a)\).

- \(h^{ab}, t_{ab}\) - degenerate metrics with signatures \((0, 1, 1, 1)\) and \((1, 0, 0, 0)\).
- \(h^{ab}t_{ab} = 0\)  (*orthogonality*)
- \(\nabla_c h^{ab} = 0 = \nabla_c t_{ab}\)  (*compatibility*) \(\Rightarrow\) fails to uniquely determine \(\nabla_a\)

- Unique way exists to separate time from space:

\[\begin{align*}
\text{Any } O \text{ and } O' \text{ agree on:} \\
\bullet \text{ Time interval between any two events.} \\
\bullet \text{ Spatial interval between any two simultaneous events.}
\end{align*}\]

- Symmetry group generated by \(\mathcal{L}_x h^{ab} = \mathcal{L}_x t_{ab} = 0.\)
1. NQFTs and Particles

**Arena for NQFTs:** Classical spacetime \((M, h^{ab}, t_{ab}, \nabla_a)\).

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- \(\nabla_c h^{ab} = 0 = \nabla_c t_{ab}\) (compatibility) \(\Rightarrow\) fails to uniquely determine \(\nabla_a\)

**Ex. 1:** Neo-Newtonian spacetime (**spatiotemporally flat**): \(R^a_{bcd} = 0\).
  - Symmetry group generated by \(\mathcal{L}_x h^{ab} = \mathcal{L}_x t_{ab} = \mathcal{L}_x \Gamma^a_{bc} = 0\). (*Galilei group*)

**Ex. 2:** Maxwellian spacetime (**rotationally flat**): \(R^{ab}_{cd} = 0\).
  - Symmetry group generated by \(\mathcal{L}_x h^{ab} = \mathcal{L}_x t_{ab} = \mathcal{L}_x \Gamma^{ab}_{c} = 0\). (*Maxwell group*)

**Comparison:**

- *Same* metrical structure, *different* curvature.
- Different types of NQFTs, in flat (Neo-Newtonian) and curved classical spacetimes.
1. NQFTs and Particles

**Received View on Particles:**

(A) The QFT must admit a Fock space formulation in which *local number operators* appear that can be interpreted as acting on a state of the system associated with a bounded region of spacetime and returning the number of particles in that region.

(B) The QFT must admit a *unique* Fock space formulation in which a *total number operator* appears that can be interpreted as acting on a state of the system and returning the total number of particles in that state.

*Necessary conditions for a particle interpretation:*

(Arageorgis, Earman, Ruetsche 2003; Halvorson 2007; Halvorson and Clifton 2002; Fraser 2008)
1. NQFTs and Particles

Claim 1: Conditions (A) and (B) fail in RQFTs.

Against (B) in RQFTs:

- **Problem of Privilege**: RQFTs admit unitarily inequivalent Fock space representations of their CCRs.
- Minkowski spacetime exemption? Kay (1979): Minkowski quantization is unique up to unitary equivalence.
- **But**: The Unruh Effect (in one guise) says: "No!" (at least to some authors).
- **In any event**: Haag's Theorem says "No!" for realistic (interacting) RQFTs.

**Haag's Theorem** ⇒

- Representations of the CCRs for both a non-interacting and an interacting RQFT cannot be constructed so that they are unitarily equivalent at a given time.

- Free particle total number operators cannot be used in interacting RQFTs.
- No consistent method for constructing "interacting" total number operators.
1. NQFTs and Particles

Claim 1: Conditions (A) and (B) fail in RQFTs.

Against (A) in RQFTs:

- **Separability Corollary (Streater & Wightman 2000):** Let $\mathcal{A}$ be a local algebra of operators associated with a bounded region $\mathcal{O}$ of spacetime. If
  
  (i) the vacuum state is cyclic for $\mathcal{A}$ ("local cyclicity");
  (ii) $\mathcal{O}$ has non-trivial causal complement;
  (iii) relativistic local commutativity holds;

  then the vacuum state is *separating* for $\mathcal{A}$. For any $A \in \mathcal{A}$, if $A\Omega = 0$, then $A = 0$.

- Reeh-Schlieder theorem secures (i) for Minkowski spacetime.

- Structure of Minkowski spacetime secures (ii).

- RQFTs satisfy (iii).

Thus: Annihilation operators, hence number operators, cannot be defined in $\mathcal{A}$ for RQFTs in Minkowski spacetime.
1. NQFTs and Particles

To what extent does the Separability Corollary hold for RQFTs in Lorentzian spacetimes in general?

  
  As soon as a classical field satisfies a certain hyperbolic partial differential equation, a state over the field algebra of the quantized theory, which is a ground- or KMS-state with respect to the group of time translations, has the Reeh-Schlieder property \([i.e., \text{local cyclicity}]. \) (Strohmeier 2000, pg. 106.)

- Is local cyclicity a generic feature of globally hyperbolic Lorentzian spacetimes?

- If so, then local cyclicity is not a generic feature of RQFTs in Lorentzian spacetimes:
  - Global hyperbolicity is not a necessary condition for the existence of an RQFT in a Lorentzian spacetime. (Fewster and Higuchi 1996.)
1. NQFTs and Particles

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- Is local cyclicity a generic feature of states analytic in the energy?

- Perhaps for RQFTs in Lorentzian spacetimes, but not for NQFTs in classical spacetimes:
  - Vacuum states for NQFTs are analytic but not locally cyclic for local algebras defined on spatial regions.
1. NQFTs and Particles

Claim 2: Conditions (A) and (B) hold in NQFTs due to the absolute temporal metric of classical spacetimes.

Condition (A) in NQFTs:

- Non-relativistic local commutivity \(\Rightarrow\) distinction between spatiotemporal local algebras and spatial local algebras.

- For spatiotemporal local algebra:
  - Requardt (1982) \(\Rightarrow\) Vacuum is locally cyclic.
  - But: Absolute temporal structure \(\Rightarrow\) Causal complement of \(\mathcal{O}\) is trivial.
  - Hence: Vacuum is not separating.

- For spatial local algebras:
  - No local cyclicity result.
  - Hence: Vacuum is not separating.
1. NQFTs and Particles

Why does local cyclicity fail for local algebras associated with spatial regions of a classical spacetime?

- Let $\phi(t, x)$ be a positive-frequency solution to a well-posed PDE.
  - $\phi(t, x)$ is a boundary value of a holomorphic function.
- Let $S$ be an open spatial region of spacetime.
  - If $\phi(t, x)$ vanishes on $S$, then it vanishes in $D(S)$.

- **Case 1:** Hyperbolic PDE in Lorentzian spacetime.
  - $D(S)$ has non-zero temporal extent.
  - If $\phi$ vanishes on $S$, then it vanishes in an open set in time, and thus everywhere (Edge of the Wedge theorem).
  - *Thus:* If $\phi \neq 0$, then it cannot vanish on $S$. *Anti-locality for spatial regions.*

- **Case 2:** Parabolic PDE in classical spacetime.
  - $D(S)$ has zero temporal extent.
  - If $\phi$ vanishes on $S$, then it need not vanish in an open set in time.
  - *Thus:* If $\phi \neq 0$, then it can vanish on $S$. *Anti-locality fails for spatial regions.*
1. NQFTs and Particles

**Claim 2**: Conditions (A) and (B) hold in NQFTs due to the absolute temporal metric of classical spacetimes.

**Condition (B) in NQFTs:**

- **No Problem of Privilege**: The absolute temporal metric guarantees a unique global time function on the spacetime, and this guarantees a unique means to construct a one-particle structure over the classical phase space (barring topological mutants).
1. NQFTs and Particles

**General Moral:**

To the extent that Conditions (A) and (B) require the existence of an absolute temporal metric, they are informed by a non-relativistic concept of time, and thus are inappropriate in informing interpretations of RQFTs.
2. **Newtonian Quantum Gravity**

I. Theories of Newtonian Gravity (NG) with a grav. potential field $\Phi$. 

$(M, h^{ab}, t_{ab}, \nabla a, \Phi, \rho)$

$$h^{ab}t_{ab} = 0 = \nabla c h^{ab} = \nabla c t_{ab}$$

Orthogonality/compatibility

$$h^{ab}\nabla a \nabla b \Phi = 4\pi G \rho$$

Poisson equation

$$\xi^a \nabla a \xi^b = -h_{ab} \nabla a \Phi$$

Equation of motion

**Ex. 1:** Neo-Newtonian NG

$$R^a_{bcd} = 0$$

**Ex. 2:** "Island Universe" Neo-Newtonian NG

$$R^a_{bcd} = 0, \quad \Phi \rightarrow 0 \text{ as } x^i \rightarrow \infty$$

**Ex. 3:** Maxwellian NG

$$R^{ab}_{\ cd} = 0$$
2. Newtonian Quantum Gravity

II. Theories of Newton-Cartan Gravity (NCG) that subsume $\Phi$ into connection. $(M, h^{ab}, t_{ab}, \nabla_a, \rho)$

\[ h^{ab} t_{ab} = 0 = \nabla_c h^{ab} = \nabla_a t_{ab} \quad \text{Orthogonality/compatibility} \]

\[ R_{ab} = 4\pi G \rho t_{ab} \quad \text{Generalized Poisson equation} \]

\[ \xi^a \nabla_a \xi^b = 0 \quad \text{Equation of motion} \]

**Ex. 1:** Weak NCG (1/$c$ → 0 limit of GR)

\[ R_{[a}^{\ [b} c_{d]} = 0 \]

**Ex. 2:** Asymptotically spatially flat weak NCG (recovers Poisson equ.)

\[ R_{[a}^{\ [b} c_{d]} = 0, \quad R^{abcd} = 0 \text{ at spatial infinity} \]

**Ex. 3:** Strong NCG (recovers Poisson equ.)

\[ R_{[a}^{\ [b} c_{d]} = 0, \quad R^{ab}_{\ ;cd} = 0 \]
2. Newtonian Quantum Gravity

**Strong NCG**

- Christian (1997): constrained Hamiltonian system, reduced phase space.
- Unique one-parameter family of time evolution maps ⇒ Unique Fock space quantization

**Newtonian Quantum Gravity (NQG)**

- Interacting (extended) Maxwell-invariant QFT of gravity in curved classical spacetime ("strong Newton-Cartan" spacetime).
- Satisfies Conditions (A) and (B).
- Gravitational degrees of freedom are *dynamic*: Compare with RQFTs in curved Lorentzian spacetimes.
- Gravitational degrees of freedom are *quantized*: Compare with semi-classical quantum gravity.
3. Intertheoretic Relations

\[ \frac{1}{c} \rightarrow 0 \text{ limit} \]

- Contraction of Poincaré Group? (Bacry & Levy-Leblond 1968)
- SR → CM, RQFT → GQM: Depends on dynamics. (Brown & Holland 2003)
- GR → NCG: No.
3. Intertheoretic Relations

$G \to 0$ limit: Ricci vs Riemann flatness

- GR $\to$ SR: Vacuum Einstein spacetime vs Minkowski spacetime
- NCG $\to$ CM, NQG $\to$ GQM: Ricc-flat classical spacetime vs Neo-Newtonian spacetime

Christian (1997)
\( \hbar \to 0 \) limit: \textit{Problem of Privilege}

- RQFT \( \to \) SR: No unique (up to unitary equivalence) representation of CCRs.
- GQM \( \to \) CM, NQG \( \to \) NCG: No problem (barring topological mutants).

Christian (1997)
3. Intertheoretic Relations

**Structural Problem**

- What is the referant of "GQM"? Where do NQFTs fit in?

**Proposal:** Add another axis for $N = degrees of freedom$

- Let "NQM" refer to non-relativistic finite-dimensional quantum theories of particle dynamics.
- Consider NQMs to be the $N \to 0$ limit of NQFTs.

Christian (1997)
3. Intertheoretic Relations

- Particle vs field theories ($N$ axis).
- Relativistic vs non-relativistic theories ($1/c$ axis).
- Gravitational vs non-gravitational theories ($G$ axis).
- Classical vs quantum theories ($\hbar$ axis).
3. Intertheoretic Relations

Turning off $G$ in field theories:

- Non-relativistic classical field theory of gravity $\rightarrow$ NCFT
- Asymptotically spatially flat NCG = "Island Universe" Neo-Newtonian NG
- $G \rightarrow 0$: Galilei-invariant classical field theory in Neo-Newtonian spacetime
Turning off $G$ in field theories:

- Relativistic classical field theory of gravity $\rightarrow$ RCFT
- GR
- $G \rightarrow 0$: Relativistic classical field theory in Ricci-flat Lorentzian spacetime
3. Intertheoretic Relations

Turning off $G$ in field theories:
- Non-relativistic quantum field theory of gravity $\rightarrow$ NQFT
- NQG
- $G \rightarrow 0$: NQFT in Ricci-flat classical spacetime
3. Intertheoretic Relations

**Turning on quantum gravity:**
- Quantizing GR.
3. Intertheoretic Relations

**Turning on quantum gravity:**
- Quantizing GR.
- Turning on gravity in an RQFT.
3. Intertheoretic Relations

Turning on quantum gravity:

- Quantizing GR.
- Turning on gravity in an RQFT.
- Relativizing NQG.
3. Intertheoretic Relations

Turning on quantum gravity:
- Quantizing GR.
- Turning on gravity in an RQFT.
- Relativizing NQG.
- Taking the "thermodynamic limit" of an RQMG.