14: Generalized Elements in $S^\circ$

Structure-preserving maps from a cycle to another endomap

Let $X^\alpha$ and $Y^\beta$ be the $S^\circ$-objects (i.e., dynamical systems):

Find an $S^\circ$-map $X^\alpha \longrightarrow Y^\beta$ such that $f(0) = y$.

To specify $f$, we need to say what it does to each element $x$ of $X^\alpha$:

$x = 0$: \hspace{2cm} \begin{align*}
\alpha(0) &= \beta(f(0)), \text{ or } f(1) = \beta(y) = z \\
\alpha(1) &= \beta(f(1)), \text{ or } f(2) = \beta(z) = y \\
\alpha(2) &= \beta(f(2)), \text{ or } f(3) = \beta(y) = z \\
\alpha(3) &= \beta(f(3)), \text{ or } f(0) = \beta(z) = y
\end{align*}

How many other maps are there? (Only other one takes 0 to z.)

Definition: An element $x$ of an $S^\circ$-object $X^\alpha$ has period $n$ just when $\alpha^n(x) = x$.

Definition: For any natural number $n$, the cycle of length $n$, $C_n$, is the set of $n$ elements \{0, 1, 2, ..., $n$\} with the "successor" endomap, with the successor of $n - 1$ being 0.
Note: \( S^{\circ} \)-maps \( C_4 \xrightarrow{f} Y^{\circ^3} \) correspond to all elements of \( Y^{\circ^3} \) with period 4!

\[
Y^{\circ^3} = \begin{array}{c}
\text{Two elements with period 4} \\
y: \beta^4(y) = y \\
z: \beta^4(z) = z
\end{array}
\]

\[
\text{Two maps } C_4 \xrightarrow{f} Y^{\circ^3}: \\
f_1(0) = y, f_1(1) = z, f_1(2) = y, f_1(3) = z \\
f_2(0) = z, f_2(1) = y, f_2(2) = z, f_2(3) = y
\]

In general: For any arbitrary \( S^{\circ} \)-object \( Y^{\circ^3} \), the \( S^{\circ} \)-maps \( C_n \xrightarrow{f} Y^{\circ^3} \) correspond to all elements of \( Y^{\circ^3} \) with period \( n \).

Terminology: The \( S^{\circ} \)-maps \( C_n \xrightarrow{f} Y^{\circ^3} \) name the elements of \( Y^{\circ^3} \) with period \( n \).

Question: How can we name arbitrary or "generalized" elements of an \( S^{\circ} \)-object?

example 1:

\[
Y^{\circ^3} = \begin{array}{c}
y \rightarrow \beta^4(z) = z; \text{ so } z \text{ has period 4} \\
x \text{ has no period; but } x \text{ has the "positive property" of "being two steps away from a 4-cycle"}
\end{array}
\]

example 2:

\[
N^{\circ^\sigma} = \begin{array}{c}
\text{set of natural numbers } \{0, 1, 2, 3, \ldots\} \\
\sigma(n) = n + 1
\end{array}
\]

0 has no positive properties
Claim: \( S^{\sigma}\)-maps from \( N^{\sigma} \) to any \( S^{\sigma}\)-object \( Y^{\sigma} \) name all the elements of \( Y^{\sigma} \).

In particular: For each element \( y \) of \( Y^{\sigma} \), there is a unique map \( N^{\sigma} \xrightarrow{f} Y^{\sigma} \) such that \( f(0) = y \).

Proof. Let \( N^{\sigma} \xrightarrow{f} Y^{\sigma} \) be an \( S^{\sigma}\)-map such that \( f(0) = y \) for element \( y \) of \( Y^{\sigma} \).

Now: Show that any other \( S^{\sigma}\)-map \( N^{\sigma} \xrightarrow{g} Y^{\sigma} \) is such that, if \( g(0) = f(0) \), then \( g = f \).

Given:

<table>
<thead>
<tr>
<th>Given</th>
<th>( f \circ \sigma = \beta \circ f )</th>
<th>( g \circ \sigma = \beta \circ g )</th>
<th>( g(0) = f(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( N \xrightarrow{f} Y )</td>
<td>( N \xrightarrow{g} Y )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \sigma \downarrow \beta )</td>
<td>( \sigma \downarrow \beta )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( f(0) )</td>
<td>( g(0) )</td>
<td></td>
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</tbody>
</table>

Then:

\[
\begin{align*}
\text{Then:} & \quad f(1) = f(\sigma(0)) \quad \text{given (definition of } \sigma) \\
& = \beta(f(0)) \quad \text{given (1)} \\
& = \beta(g(0)) \quad \text{given (3)} \\
& = g(\sigma(0)) \quad \text{given (2)} \\
& = g(1) \quad \text{given (definition of } \sigma) \\
\end{align*}
\]

So: If \( f \) and \( g \) agree on \( 0 \), then they agree on \( 1 \).

Now: Suppose for any \( n \), \( g(n) = f(n) \). Call this assumption \( (3') \).

Does this then entail \( f(n + 1) = g(n + 1) \)?

Check:

\[
\begin{align*}
\text{Check:} & \quad f(n + 1) = f(\sigma(n)) \quad \text{given (definition of } \sigma) \\
& = \beta(f(n)) \quad \text{given (1)} \\
& = \beta(g(n)) \quad \text{given (3')} \\
& = g(\sigma(n)) \quad \text{given (2)} \\
& = g(n + 1) \quad \text{given (definition of } \sigma) \\
\end{align*}
\]

So: We’ve shown that if \( f(0) = g(0) \), then \( f(1) = g(1) \). And if \( f(n) = g(n) \) for any \( n \), then \( f(n + 1) = g(n + 1) \). This means, if \( f \) and \( g \) agree on \( 0 \), then they agree on \( 1 \), and hence \( 2 \), and hence \( 3 \), etc. So they agree on all elements of \( N \).

So: \( f = g \)!