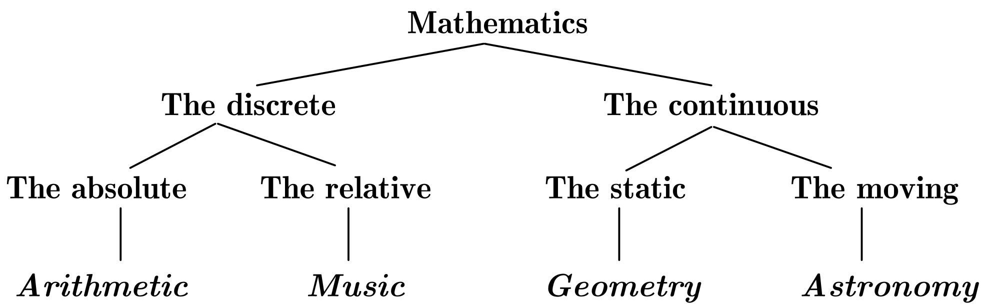
Course Intro

Topics

I. The Branches of Mathematics

According to the Greeks...



Middle Ages add Grammar, Rhetoric, Logic to form the "Seven Liberal Arts"

According to the Geeks...

1. Algebra
- Abstract algebra
  - Theory of groups, rings, fields, algebras, modules, vector spaces, *etc.*
- Combinatorics
- Number Theory

2. Analysis
- Calculus
- Real and Complex Analysis
- Vector and Tensor Analysis
- Differential Equations
- Functional Analysis

3. Geometry
- Euclidean and Non-Euclidean Geometry
- Affine, Metric, Projective Geometry
- Discrete Geometry
- Differential Geometry
- Algebraic Geometry

4. Applied Mathematics
- Probability
- Statistics
- Game Theory
- Systems and Control Theory
- Computer Science

5. Foundations
- Logic, Computability, Recursion Theory
- Set Theory
- Category Theory

Central to all branches:

Numbers *What are they?*

Most basic type: 1, 2, 3, 4, 5, ...

BUT: Many other types!

ASIDE: Philosophers like to be pedantic and make the distinction between *numbers*, which are *concepts* of some sort, and *numerals*, which are *symbols* we use to represent numbers. Thus the symbols 1, 2, 3, 4, ... are numerals that represent the numbers one, two, three, four, *etc.* Just as I, II, III, IV, are (Roman) numerals that represent the same numbers. We won't be so pendantic in our usage of the term "number".

II. A Beastiary of Number Systems

Natural numbers. $\mathbb{N} = \{0, 1, 2, 3, ...\}$ *zero!*

Integers. $\mathbb{Z} = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$ *negative numbers!*

Rational numbers. $\mathbb{Q} = \left\{ \begin{array}{l} \text{all numbers that can be written} \\ \text{as the ratio of two integers} \end{array} \right\}$

$= \{\text{all } p/q, \text{ where } p,\ q \in \mathbb{Z}, \text{ and } q \neq 0\}$

Can't divide by 0:
If we could, then since
$n \times 0 = m \times 0$
for any numbers n, m,
we would have $n = m$
for any n, m.

examples:

$1/2 = 0.5000...$
$1/3 = 0.3333...$
$1/7 = 0.142857142857...$

decimal expansions have repeating patterns

Irrational numbers $= \left\{ \begin{array}{l} \text{all numbers that } \textit{cannot} \text{ be written} \\ \text{as the ratio of two integers} \end{array} \right\}$

examples:

$\sqrt{2} = 1.4142135...$
$\pi = 3.141592653...$
$e = 2.7182818284...$

decimal expansions do not have repeating patterns

Real numbers. $\mathbb{R} = \{\text{rational numbers and irrational numbers}\}$

Complex numbers. $\mathbb{C} = \{\text{all } p + iq, \text{ where } p,\ q \in \mathbb{R}, \text{ and } i^2 = -1\}$

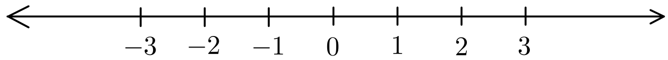
"imaginary" number!

Complex multiplication

$$\begin{aligned}(2 + i3) \times (1 + i6) &= (2 \times 1) + (2 \times i6) + (i3 \times 1) + (i3 \times i6) \\ &= 2 + i12 + i3 - 18 \\ &= -16 + i15\end{aligned}$$

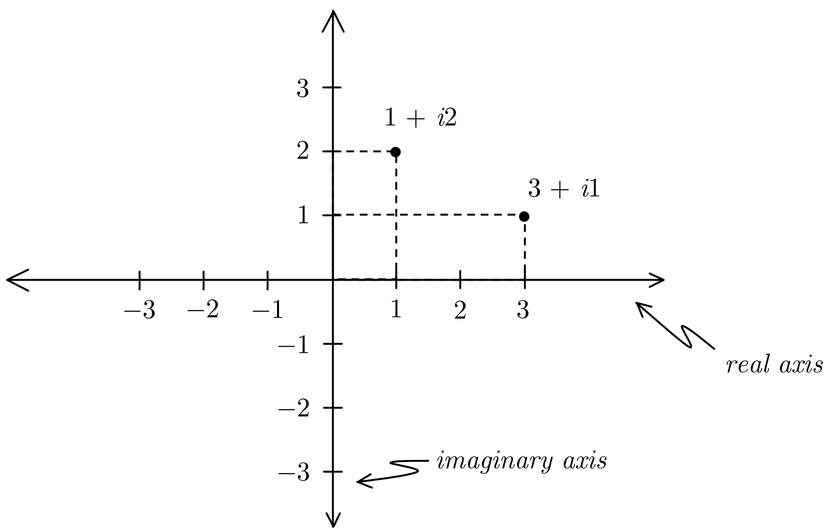
Geometric representations

$\mathbb{R}$ - "1-dimensional" number system. Can be represented by a line:



The real number line: Points on a 1-dim line correspond to real numbers.

$\mathbb{C}$ - "2-dimensional" number system. Can be represented by a plane:



The complex plane: Points on a 2-dim plane correspond to complex numbers.

Higher dimensional number systems

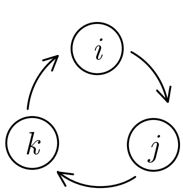
Quaternions.

(4-dim number system: Can represent as points on a 4-dim "hypercube")

$$\mathbb{H} = \left\{ \begin{array}{l} \text{all } p + iq + jr + ks, \text{ where } p,\ q,\ r,\ s \in \mathbb{R}, \text{ and} \\ i^2 = j^2 = k^2 = -1 \\ ij = k, \qquad ji = -k \\ jk = i, \qquad ik = -j \\ ki = j, \qquad kj = -i \end{array} \right\}$$

multiplication table for i, j, k

3 different types of imaginary numbers



reversing arrows corresponds to multiplying by −1

Quaternionic multiplication

$$(2 + i3 + j5 + k2) \times (1 + i2 + j2 + k5) = 2 + i4 + j4 + k10 + i3 + i^2 6 + ij6 + ik15 + j5 + ji10 + j^2 10 + jk25 + k2 + ki4 + kj4 + k^2 10$$

$$= 2 + i4 + j4 + k10 + i3 - 6 + k6 - j15 + j5 - k10 - 10 + i25 + k2 + j4 - i4 - 10$$

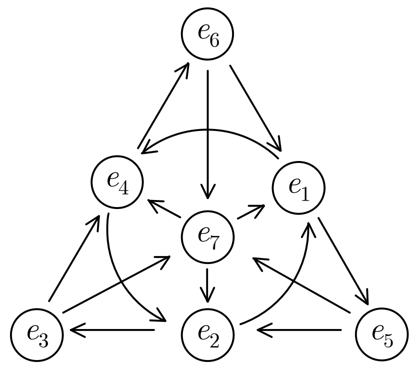
$$= -24 + i28 - j2 + k8$$

Octonions.

(8-dim number system: points on an 8-dim hypercube)

$\mathbb{O} = \{$ all $p + e_1 q + e_2 r + e_3 s + e_4 t + e_5 u + e_6 v + e_7 w$, where $p, q, r, s, t, u, v, w \in \mathbb{R}$, and $e_1{}^2 = e_2{}^2 = e_3{}^2 = e_4{}^2 = e_5{}^2 = e_6{}^2 = e_7{}^2 = -1$, with $\}$

7 different types of imaginary numbers



Multiplication Rules for the imaginary numbers. Each set of three numbers connected by three arrows represents a cyclical rule (like the rules for the quaternions). (The vertices of the outer triangle are implicitly connected by arrows.) Reversing arrows corresponds to multiplying by −1.

Sedonions. $\mathbb{S}$ = 16-dim number system, with 15 different types of imaginary numbers!

Subtleties:

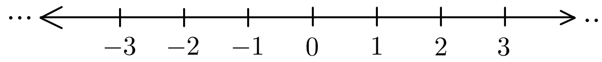
- The number systems from $\mathbb{R}$ on up form *algebras*. They consist of objects (real numbers and imaginary numbers) with rules for how to add and multiply them. These rules place constraints on the dimension of the system.

- Even more subtle: For all of these number systems, how large are they? How *many* numbers do they contain?

Natural numbers. $\mathbb{N} = \{0, 1, 2, 3, ...\}$

How many natural numbers are there?

Real numbers. $\mathbb{R} =$



How many real numbers are there? (How many points are on a line?)

<u>**Initial Response**</u>: An <u>*infinite*</u> amount!

What's this?

∞ "infinity"

<u>Story to come:</u>

- Historical development of concept of infinity: from early Greeks to the Calculus to 19th century set theory.

- *One result*: The infinity of real numbers is greater than the infinity of natural numbers!

- *Another result*: Development of "transfinite" number systems: types of numbers that go beyond the infinities of the natural and real numbers!

- *Yet another result*: Attempt to base all branches of mathematics on set theory. Sets as fundamental objects of mathematics (more fundamental than numbers!).

- *More recently*: "Categories" (*objects* and *arrows*) as more fundamental than sets and numbers.